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Mathematical control of the saltwater intrusion in coastal aquifers.

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Introduction

• The freshwater, is a scarce commodity and extremely useful to the human life.

• Only 3% of the quantity of water in earth is fresh.

• The aquifers are, for most of the countries, a source of supply in freshwater.

• Concentrated populations in the Coastal areas result in increased demand for freshwater and accelerated groundwater pumping.

• Water needs for agriculture, industry and public water supplies, groundwater resources have been seriously over exploited.

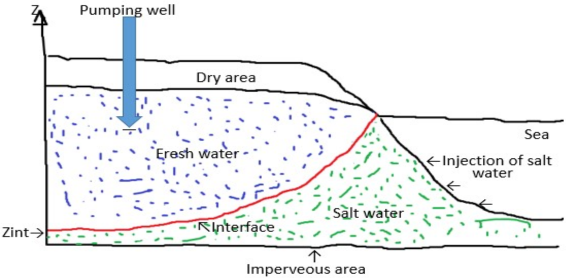
• Globally, fresh groundwater resources in coastal aquifers are significantly impacted by seawater intrusion.

• Seawater intrusion in coastal aquifers is a common problem and is encountered, with different degrees, in almost all coastal aquifers.

• It is regarded as a natural process that might be accelerated or retarded by external factors such as increase or decrease in the groundwater pumping, irrigation and recharge practices, land use, and possible seawater rise due to the impacts of global warming.

- A comprehensive review on different aspects of seawater intrusion assessment, monitoring, and modeling is provided by Bear and al.
- Physically, seawater intrusion is a density-dependent problem.
- Modeling a seawater intrusion process needs to couple groundwater flow equation with solute (salt) transport equation.
- It is based on the groundwater flow field, which is in turn affected by salt and density distribution in the groundwater field.

Model description



To obtain our model, we consider the continuity equations coupled with Darcy's equation, means mass conservation laws for each phase (fresh and salt water) coupled with the classical Darcy law for porous media. The two phases are the same fluid with different caracteristics. The flow is then governed by the same laws. Therefore, we consider just one phase to obtain the model and for the second phase, the model is obtained by similarity.

The domaine $\Omega \subset \mathbb{R}^3$ composed by :

Ω_s, the salt water phase, and

Ω_f, the fresh water phase.

Between the two phases, we have the interface.

Local models

Let θ_s the saltwater content, ρ_s the density and U_s the velocity. The continuity equation is given by

$$div\,(\rho_s U_s) + \frac{\partial(\rho_s \theta_s)}{\partial t} = \rho_s(q_s^p - q_s^t) \tag{1}$$

where q_s^p (respectively q_s^t) is the provided mass flow (respectively the taken mass flow).
The effective velocity U_s of the flow is thus related to the pressure P through the so-called Darcy law

$$U_s = -\frac{k}{\mu}\left(\nabla p_s + \rho_s g \nabla z_s\right), \tag{2}$$

where ρ_s and μ are respectively the density and the viscosity of the fluid, k is the permeability of the soil and g the gravitational acceleration constant.
Introducing the piezometric head h_s defined by

$$h_s = \frac{p_s}{\rho_s g} + z_s \tag{3}$$

where z_s is the elevation of the considering particle, we write equation (2) as follows

$$U_s = -K \nabla h_s, \tag{4}$$

where $K = \dfrac{k \rho_s g}{\mu}$ is the hydraulic conductivity which expresses the ability of the underground to conduct the fluid.

The saltwater content, given by $\theta_s = \dfrac{V_s}{V}$, where V_s is the volume of Ω_s and V is the volume of Ω, verifies

$$d\theta_s = \frac{1}{V}(1-\theta_s)\, dV_s. \tag{5}$$

We assume that the solid matrix of the aquifer is non-deformable. However, to take into account the contribution of its compressibility effect in the specific storage, we will assume that the solid matrix is elastic, i.e. there is a linear relationship between the effective compressive stress and the strain.

The state equation of the subdomain Ω_s is then given by

$$\frac{\partial \theta_s}{\partial t} = \frac{V_s}{V}(1 - \theta_s)\alpha \frac{\partial p_s}{\partial t}, \tag{6}$$

where p_s and α are respectively the pressure of salt water and the specifique coefficient of compressibility of the media. The partial derivative of the pressure p_s according the time is given by the following equation : From equation (3), we have $p_s = \rho_s g(h_s - z_s)$, therefore

$$\frac{\partial p_s}{\partial t} = g(h_s - z_s)\frac{\partial \rho_s}{\partial t} + \rho_s g \frac{\partial h_s}{\partial t}. \tag{7}$$

Combining the state equation of the salt water given as follow

$$\frac{\partial \rho_s}{\partial t} = \rho_s \beta_s \frac{\partial p_s}{\partial t}. \tag{8}$$

and equation (7) we deduce this following relation :

$$[1 + (z_s - h_s)\rho_s \beta_s g] \frac{\partial p_s}{\partial t} = \rho_s g \frac{\partial h_s}{\partial t}. \tag{9}$$

with β_s the compressibility coefficient of the salt water. And since $(z_s - h_s)\rho_s \beta_s g << 1$, we obtain

$$\frac{\partial p_s}{\partial t} = \rho_s g \frac{\partial h_s}{\partial t}. \tag{10}$$

Developing the continuity equation (1) of the salt water we obtain :

$$\rho_s div(U_s) + (U_s \cdot \nabla)\,\rho_s + \rho_s \frac{\partial \theta_s}{\partial t} + \theta_s \frac{\partial \rho_s}{\partial t} \quad = \quad \rho_s(q_s^p - q_s^t) \qquad (11)$$

Since $\rho_s \neq 0$ and does not depend on the space variable, with relations (6) and (8), equation (11) becomes

$$div(U_s) + \frac{V_s}{V}(1-\theta_s)\alpha \frac{\partial p_s}{\partial t} + \theta_s \beta_s \frac{\partial p_s}{\partial t} \quad = \quad (q_s^p - q_s^t).(12)$$

Considering equation (10) and setting the specific storage coefficient of Ω_S

$$S_s \quad = \quad \rho_s \theta_s g \left[\beta_s + \frac{V_s}{V} \frac{1-\theta_s}{\theta_s} \alpha \right], \qquad (13)$$

equation (12) becomes

$$div(U_s) + S_s \frac{\partial h_s}{\partial t} = (q_s^p - q_s^t). \quad (14)$$

With the Darcy's equation in each phase, we obtain the following system in the salt water phase

$$\left\{ \begin{array}{rcl} div(\vec{U_s}) + S_s \frac{\partial h_s}{\partial t} & = & q_s^p - q_s^t \;\text{ in }\; [0,T] \times \Omega_s, \\ \vec{U_s} & = & -K_s \nabla h_s. \end{array} \right. \quad (15)$$

and by similarity the governing system for the fresh water movement

$$\left\{ \begin{array}{rcl} div(\vec{U_f}) + S_f \frac{\partial h_f}{\partial t} & = & q_f^p - q_f^t \;\text{ in }\; [0,T] \times \Omega_d, \\ \vec{U_f} & = & -K_f \nabla h_f. \end{array} \right. \quad (16)$$

where S_f is given like in (13) replacing the 's' indice by 'f'.

Being in the same homogeneous media Ω, we take $K_s = K_f = k$. The specific storage coefficient of the salt water and the fresh water, S_s and S_f respectively, depend of θ_s and θ_f respectively. To close the models (15) and (16), we consider the case S_s and S_f are constants.

Global model

Now we need to find a valid system throughout the domaine Ω, means a global model of the phenomena. For that, we set global variables using indicator function which is defined as follows :

$$\chi_a(x) = \begin{cases} 1 & \textbf{if } x \in \Omega_a \\ 0 & \textbf{elsewhere} \end{cases} \tag{17}$$

like

$$\left\{ \begin{array}{lcl} h & = & \chi_f h_f + \chi_s h_s, \\ S & = & \chi_f S_f + \chi_s S_s, \\ U & = & \chi_f U_f + \chi_s U_s \\ q^p & = & \chi_f q_f^p + \chi_s q_s^p, \\ q^t & = & \chi_f q_f^t + \chi_s q_s^t. \end{array} \right. \tag{18}$$

This interface is then considered like a contact surface ; that means there is continuity of the pressure at the interface.

The piezometric head in the salt water, h_s and in the fresh water, h_f are given respectively by

$$h_s = \frac{p_s}{\rho_s g} + z_s \quad \text{and} \quad h_f = \frac{p_f}{\rho_f g} + z_f .$$

Let a particle at the interface with elevation z_{int}, means $z_s = z_f = z_{int}$, with the continuity of the pression at the interface, $p_s = p_f$, we have

$$\rho_s g(h_s - z_{int}) = \rho_f g(h_f - z_{int})$$

$$\text{means} \quad z_{int} = \frac{\rho_s}{\rho_s - \rho_f} h_s - \frac{\rho_f}{\rho_s - \rho_f} h_f .$$

We obtain the following relation

$$z_{int} = -\delta h_f + (1+\delta) h_s, \quad \text{with} \quad \delta = \frac{\rho_f}{\rho_s - \rho_f}. \qquad (19)$$

Only pumping freshwater throughout the pumping well ω is considered, and let q_f, the pumping term. We assume that there are not providing freshwater and not taking saltwater, i.e. $q_f^p = 0$ and $q_s^t = 0$. Let Γ_s the part of the boundary of Ω_s where the saltwater injection is done and let q_s the injection term.

Considering systems (15) and (16) and the expression of z_{int} given in (19), we have the following equation

$$S\frac{\partial z_{int}}{\partial t} - k\Delta z_{int} = -\delta(q_f^p - q_f^t) + (1+\delta)(q_s^p - q_s^t) \quad (20)$$

where S is given in (18). Like in the local models, we study only the case $S_s = S_f = S = constant$.
Under the pumping of freshwater and the injection of saltwater, we obtain the following system governing the movement of the interface.

$$\left\{ \begin{array}{rcll} S\frac{\partial z_{int}}{\partial t} - k\Delta z_{int} & = & \delta q_f^p \mathbf{1}_w & \text{in } [0,T]\times\Omega, \\ \frac{\partial z_{int}}{\partial n} & = & (1+\delta)q_s^t & \text{on } [0,T]\times\Gamma_s, \\ z_{int}(0,x,y) & = & z_{int,0}(x,y) & \text{in } \Omega. \end{array} \right. \quad (21)$$

Numerical simulation

For the numerical simulation of our model, we use the P1 Lagrange finite element to deal with the spatial discretization of the problem (21). For that, it is convenient to write the variational formulation of the problem.

$$
\begin{aligned}
k\int_\Omega \nabla z_{int}\nabla v dX + S\int_\Omega \frac{\partial}{\partial t} z_{int}.v dX & \\
-k\int_{\Gamma_s}(1+\delta)q_s v d\sigma &= \int_\Omega \delta(q_f)\mathbf{1}_w v dX, \\
z_{int}(0,x) &= z_{int,0}(x).
\end{aligned}
\tag{22}
$$

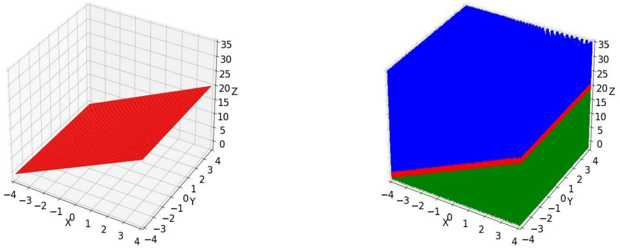
The time operator $\dfrac{\partial}{\partial t} z_{int}$ is approximated by an implicit Euler scheme

$$\frac{\partial z_{int}}{\partial t} = \frac{z_{int}^{m+1} - z_{int}^{m}}{dt}.$$

The variational formulation is given as follow

$$\begin{array}{rcl} \displaystyle\int_{\Omega} (S z_{int}^{m+1} v + dt.k \nabla z_{int}^{m+1} \nabla v) dX & & \\ -dt.k \int_{\Gamma_s} (1+\delta) q_s v d\sigma & = & \displaystyle\int_{\Omega} (S z_{int}^{m} v \\ +dt.\delta(q_f) \mathbf{1}_w v) dX, & & \\ z_{int}(0,x) & = & z_{int,0}(x). \end{array} \tag{23}$$

The softwares are FreeFem++ for the resolution of the model and Python for visualization of the obtained results.
The initial state of our domain is illustrated in figures 2 where we have the interface alone in (a) and the interface and the two fluids (freshwater and saltwater) in (b).



(a) (b)

FIGURE – Schematic representation of the aquifer at the initial state. Data : (a) the full interface, (b) the whole aquifer with the saltwater in green, the freshwater in blue and in red the interface.

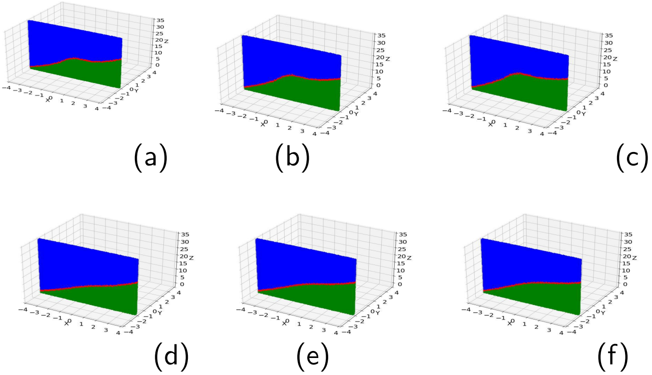


FIGURE – Data : (a), (b), and (c) are a vertical section at $y = -1$, (d), (e), and (f) at $y = -2$ for pumping duration t= 40, 50 and 60

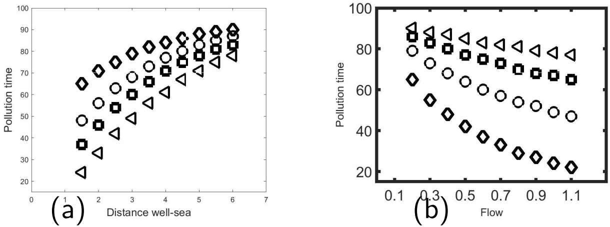


FIGURE – Pumping flow and distance well-sea effect on the pollution time. Data : (a) Pollution time versus distance well-sea for fixed flows given by the well radius ; ($\diamond$) $r = 0.2$, ($\circ$) $r = 0.4$, ($\square$) $r = 0.6$, ($\triangleleft$) $r = 1$; (b) Pollution time versus the flow for different fixed distance, d, between the well and Γ_s ; ($\diamond$) $d = 1.5$, ($\circ$) $d = 3$, ($\square$) $d = 4.5$, ($\triangleleft$) $d = 6$.

Optimal control results

We made our study on a finite time interval $[0, T]$, T a real strictly positive. The state variable is :

$z_{int} \ : \ [0, T] \times \Omega \longrightarrow \mathbb{R}$.

The initial state : $z_{int,0} \ : \ \Omega \longrightarrow \mathbb{R}$.

Our pollution threshold : $z_{sp} \ : \ \Omega \longrightarrow \mathbb{R}$**, this is a surface located at a higher level than our initial interface** $z_{int,0}$**.**

Internal control

We consider as control element : $q_f^P \; : \; [0,T] \times w \longrightarrow \mathbb{R}$**.**

And we define our admissible set, denoted U_{ad} **by**

$$U_{ad} = \{ q_f^P \in C^1([0,T], L^2(w)), \; : \; \delta q_f^P(t,x,y) \in]0, Q_f], \quad \forall (t,x,y) \in [0,T] \times w \}$$

Where Q_f **is a real strictly positive and large enough. Remark that** U_{ad} **is not empty because any positive constant and less than** $\frac{Q_d}{\delta}$ **belongs to** U_{ad}**. We suppose that the injection flow rate** q_s^t **is constant in this part.**

Objective : Reach a position close to z_{sp} with an optimal pumping flow rate q_d^{P*} at time T.
We define our cost function by

$$J(q_f^P) = \int_0^T \int_\Omega (\delta q_f^P(t,x,y))^2 \mathbf{1}_w dXdt$$
$$+\alpha \int_\Omega |z_{int}(T,x,y) - z_{sp}(x,y)|^2 \mathbf{1}_w dX,$$
$$(z_{int}, q_f^P) \text{ satisfy } (21), \ q_f^P \in U_{ad}$$

And we formulate the following optimal control problem

$$min\{J(q_f^P), q_f^P \in U_{ad}\} \tag{24}$$

There exists a unique solution q_f^{P*} for the optimal control problem 24.

Optimality system

For all $z_{int,0}$ in $L^2(\Omega)$ and for all z_{sp}, there exists an optimal pumping flow rate q_f^{P*} in U_{ad} solution of our optimal control problem and satisfying the following optimality system

$$(\mathbf{S_{int}}) = \begin{cases} \dfrac{\partial z^*_{int}}{\partial t} - k\Delta z^*_{int} & = \delta q_f^{P*} \mathbf{1}_w \text{ in } [0,T]\times\Omega, \\ \dfrac{\partial z^*_{int}}{\partial n} & = (1+\delta) q_s^A \text{ on } [0,T]\times\partial\Omega, \\ z^*_{int}(0,x,y) & = z_{int,0}(x,y) \text{ in } \Omega, \\ \dfrac{\partial p^*}{\partial t} + \Delta p^* & = 0 \text{ in } [0,T]\times\Omega, \\ \dfrac{\partial p^*}{\partial n} & = 0 \text{ on } [0,T]\times\partial\Omega, \\ p^*(T,x,y) & = 2\alpha[z^*_{int}(T,x,y) - z_{sp}(x,y)] \text{ in } \Omega \end{cases}$$

Where z^*_{int} is the optimal trajectory associated to q_f^{P*} and p^* the solution of the associate problem associated to q_f^{P*}.

THANK FOR YOUR ATTENTION