Stability of thermoelastic system with internal delay

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Outline

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1. Delay in ODEs and PDEs
2. Delayed systems
3. A concrete delayed systems with damping
   - Delay at the second equation
   - Delay at the first equation

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2. Delayed systems
3. A concrete delayed systems with damping
   - Delay at the second equation
   - Delay at the first equation
4. Abstract delayed systems with damping (I)
   - α-β systems
   - Some related systems

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- Hale et Verduyn (1993), Pruβ (1993), Diekmann et al (1995), Chandrasekharaiah (1998), Bàtkai and Piazzera (2005) .
- All these studies show that the delay leads to instabilities (mechanical vibrations, loss of synchronization, etc.).

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70s of the 20th century: in the modeling of evolution of populations, in the progression of an epidemic or the dynamics of a cell cycle. Delay taken into account.

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- New fields of research have also emerged where DPDEs play a fundamental role (nonlinear optics, urban traffic, robotics, ...).

A simple example of ODE with delay

The following model describing the evolution of a population N

$$N'(t) = kN(t), \quad N(0) = N_0.$$

Such equation is solved by

$$N(t) = e^{kt}N(t_0).$$

The knowledge of the present (here $N(0) = N_0$) allows the prediction of the future at any time t. The past is not involved in the solution.

A simple example of ODE with delay

If we take into account the gestation period:

$$N'(t) = kN(t-\tau), \quad N(t) = N_0(t) \quad -\tau \leq t \leq 0, \qquad \text{DE} \qquad (1)$$

The delay $\tau > 0$: the gestation time ($\tau = 9$ months) for the human population.

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- For $t \in [0, \tau[$,

$$N(t) = N(0) + k \int_0^t N_0(s) ds.$$

We can then solve the system (1) on the interval $[\tau, 2\tau[$ and so on we can solve (1) on $[0, +\infty[$.

Heat equation (Fourier's law)

Heat conduction:

$$\theta_t + \gamma \, \mathrm{div} \, q = 0$$

θ: temperature , q: heat flux vector

Fourier's law,

$$q(x,t) = -k\nabla\theta(x,t)$$

Classical heat equation

$$\theta_t = k\gamma\Delta\theta.$$

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- Exponentially stable.
- Physical impossibility of infinite propagation speed.

Correction: Cattaneo's law

Morse and Feshbach, 1953; Vernotte, 1958; Cattaneo, 1958:

$$\tau \frac{\partial q}{\partial t} + q = -k \nabla \theta$$

τ represents the relaxation time.

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- Exponential stability.

Single-phase-lag constitutive law:

(Tzou 1997):

$$q(x, t + \tau) = -k\nabla\theta(x, t)$$

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- Ill-posed.
- First order approximation ($\tau = 0$): Fourier's law.
- Second order approximation: Cattaneo's law.

Two delayed PDEs

$$\theta_t(x,t) = k\gamma\Delta\theta(x,t-\tau) \quad \text{(parabolic)},$$
$$u_{tt}(x,t) = \alpha\Delta u(x,t-\tau) \quad \text{(hyperbolic)}$$

are not well-posed (Jordan et al, 2008) and (Racke et al, 2009)

- Correction: adding a non delayed term: exp. $\Delta\theta(x,t)$ (Pruβ 1993) and (Batkai and Piazzera 2005); Well-posedness.
- Correction: adding KV damping. (Ammari et al., 2015):

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$$\begin{cases} u''(t) + aBB^*u'(t) + BB^*u(t-\tau) = 0, & \text{in } (0,\infty), \\ u(0) = u_0, \quad u'(0) = u_1, & \\ B^*u(t-\tau) = f_0(t-\tau), & \text{in } (0,\tau), \end{cases}$$

where $B : \mathcal{D}(B) \subset H_1 \to H$ is a linear unbounded operator from a Hilbert space H_1 to a Hilbert space H... They obtained an exponential decay result under the assumption $\tau \leq a$.

Example

$$\begin{cases} u_{tt}(x,t) - a\Delta u_t(x,t) - \Delta u(x,t-\tau) = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,\infty), \\ \nabla u(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases}$$

$$E(t) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |u_t|^2\right) dx + \xi \int_\Omega \int_0^1 |\nabla u(x,t-\tau\rho|^2 \, d\rho dx.$$

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- If $\xi > \frac{2\tau}{a}$ and $\tau \leq a$ the energy satisfies

$$E(t) \leq Me^{-wt}E(0), \quad \forall\, t > 0$$

for some positive constants M and w.

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x, t-\tau) + \gamma\theta_x(x,t) = 0, & \text{in } (0,\ell)\times(0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, & \text{in } (0,\ell)\times(0,\infty), \\ u(0,t) = u(\ell,t) = \theta_x(0,t) = \theta_x(\ell,t) = 0, & t \geq 0 \end{cases}$$

where $\alpha, \gamma, \kappa, \ell, \tau > 0$. The functions $u = u(x,t)$ and $\theta = \theta(x,t)$ describe respectively the displacement and the temperature difference, $x \in (0,\ell)$ and $t \geq 0$.

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H is a Hilbert space.
$A : D(A) \subset H \to H$ a self-adjoint, positive definite operator.
We consider an abstract thermoelastic system with delay given by.

$$\begin{cases} u''(t) + Au(t-\tau) - A^{\beta}\theta(t) = 0, & t \in (0, +\infty), \\ \theta'(t) + A^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, & t \in (0, +\infty) \end{cases} \tag{2}$$

$$\boxed{(\beta, \alpha) \in [0, 1] \times [0, 1].}$$

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(Racke 2012)

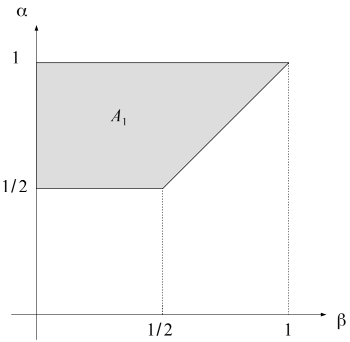


Figure: Aeria of instability A_1 (with delay at u and no damping)

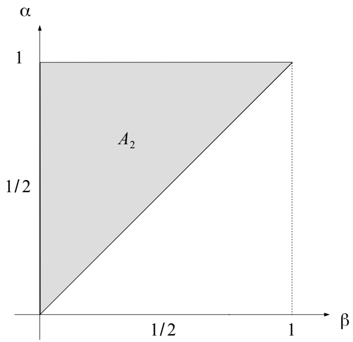


Figure: Aeria of instability A_2 (with delay at θ and no damping)

- $A_1 := \{(\beta, \alpha) \mid 0 \leq \beta \leq \alpha \leq 1,\ \alpha \geq \frac{1}{2},\ (\beta, \alpha) \neq (1,1)\}$
- $A_2 := \{(\beta, \alpha) \mid 0 \leq \beta \leq \alpha \leq 1,\ (\beta, \alpha) \neq (1,1)\}$

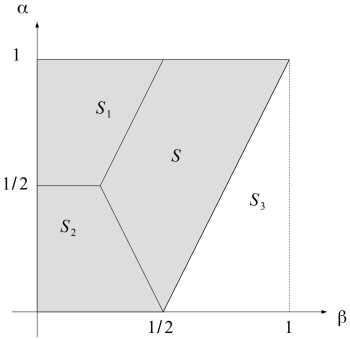
$\tau = 0$ (F. A. Khodja et *al.* 1999) and (Hao and Liu 2013)

- Well-posed: $[0,1] \times [0,1]$

  Exp. stab.: $S = \{(\beta,\alpha) \in [0,1]^2 \mid |2\beta - 1| \leq \alpha \leq 2\beta\}$

- Poly. stab.: $S_1 \cup S_2 = \{(\beta,\alpha) \in [0,1]^2 \mid 2\beta < \alpha < 1 - 2\beta\}$

  Instability: $S_3 = \{(\beta,\alpha) \in [0,1]^2 \mid 0 < \alpha < 2\beta - 1\}$.



(Mustapha and Kafini 2013)

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t) + \gamma\theta_x(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t-\tau) - a\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\ u(0,t) = u(\ell,t) = 0, & \text{in } (0,\infty), \\ \theta_x(0,t) = \theta_x(\ell,t) = 0, & \text{in } (0,\infty), \\ u_x(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & \text{in } \Omega \end{cases} \tag{4}$$

Energy of a solution of problem (4):

$$E(t) := \frac{1}{2}\int_\Omega \left(u_t^2(x,t) + \alpha u_x^2(x,t) + \theta^2(x,t)\right) dx$$
$$+\xi \int_\Omega \int_0^1 \theta_x^2(x,t-\tau\rho)\, d\rho dx.$$

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- Exponential stability for $|k| < a$ and $|k| < \xi < a$.

(Khatir and Shel 2022)

$$
\begin{cases}
u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) - \beta u_{xxt}(x,t) + \gamma\theta_x(x,t) = 0, & \text{in } \Omega \times (0,\infty), \\
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\end{cases}
\tag{5}
$$

$\Omega = (0,\ell)$. We define the energy of a solution of problem (5) as

$$
E(t) := \frac{1}{2}\int_\Omega \left(u_t^2(x,t) + \alpha u_x^2(x,t) + \theta^2(x,t)\right) dx + \xi \int_\Omega \int_0^1 u_x^2(x,t-\tau\rho)\,d\rho\,dx
$$

where $\xi > 0$ is a parameter fixed later on.

Well-posedness

We introduce, the new variable

$$z(x,\rho,t) = u_x(x,t-\tau\rho), \quad \text{in } \Omega \times (0,1) \times (0,\infty),$$

Clearly, $z(x,\rho,t)$ satisfies

$$\begin{aligned} \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \\ z(x,0,t) \;=\; u_x(x,t), \quad x \in \Omega, \quad t \in (0,+\infty). \end{aligned}$$

Well-posedness

Then, problem (5) takes the form

$$u_{tt}(x,t) - \alpha z_x(x,1,t) - \beta u_{xxt}(x,t) + \gamma \theta_x(x,t) = 0, \text{ in } \Omega \times (0,\infty),$$

$$\begin{aligned}
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad & \text{in } \Omega \times (0,1) \times (0,+\infty),\\
\theta_t(x,t) - \kappa \theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, \qquad & \text{in } \Omega \times (0,\infty),\\
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\end{aligned}$$

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z(x,\rho,0) = f_0(x,-\tau\rho), \qquad & \text{in } \Omega \times (0,1).
\end{aligned}$$

Well-posedness

Without loss of generality, we assume that $\int_\Omega \theta(x,t)dx = 0$. Let

$$\mathcal{H} = \left\{(f,g,p,h) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)) \times L^2(\Omega) \mid \int_\Omega h(x)dx = 0\right\}.$$

Equipped with the following inner product: for any $U_k = (f_k, g_k, p_k, h_k) \in \mathcal{H}, \quad k = 1, 2,$

$$\langle U_1, U_2\rangle_{\mathcal{H}} = \int_\Omega (\alpha f_{1x}(x) f_{2x}(x) + g_1(x) g_2(x) + h_1(x) h_2(x))\, dx + \xi \int_\Omega \int_0^1 p_1(x,\rho) p_2(x,\rho) d\rho dx,$$

$\mathcal{H}$ is a Hilbert space.

Well-posedness

Define

$$U := (u, u_t, z, \theta)$$

then, problem (5) can be formulated as a first order system of the form

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u_0, u_1, f_0(.,. - \tau), \theta_0) \end{cases} \quad (6)$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ (\alpha z(.,1) + \beta v_x)_x - \gamma\theta_x \\ -\frac{1}{\tau} z_\rho \\ -\gamma v_x + \kappa\theta_{xx} \end{pmatrix}$$

with domain $\mathcal{D}(\mathcal{A}) =$

$$\{U = (u, v, z, \theta) \in \mathcal{H} \cap \left[H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega; H^1(0,1)) \times H^2(\Omega)\right] \mid$$
$$\theta_x(0) = \theta_x(\ell) = 0, \quad z(.,0) = u_x \quad \text{and} \quad (\alpha z(.,1) + \beta v_x) \in H^1(\Omega)\}$$

in the Hilbert space $\mathcal{H}$.

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Lemma

If $\xi > \frac{2\tau\alpha^2}{\beta}$, then there exists $m > 0$ such that $\mathcal{A} - mId$ is dissipative maximal.

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Theorem

For any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in \mathcal{C}([0,+\infty),\mathcal{H})$ of problem (6). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U \in \mathcal{C}([0,+\infty),\mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0,+\infty),\mathcal{H})$.

Exponential stability

Based on Lyapunov method, we prove that:

Theorem

There exists $\beta_0 > 0$ such that for every $\beta \geq \beta_0$, the system (5) is exponentially stable:

$$E(t) \leq Me^{-wt}E(0), \quad \forall\, t > 0$$

for some positive constants M and w.

Recently, based on a frequency domain result for exponential stability of a $\mathcal{C}_0$ semigroup (due to Gearhard-Pruss-Huang):

Lemma

A $\mathcal{C}_0$ semigroup $e^{t\mathcal{L}}$ on a Hilbert space G satisfies

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(G)} \leq Ce^{-wt}$$

for some constants $C > 0$ and $w > 0$ if and only if

$$\mathbb{C}_0 := \{\lambda \in \mathbb{C} | \quad \mathrm{Re}(\lambda) > 0\} \subset \rho(\mathcal{L}) \tag{7}$$

and

$$\sup_{\mathrm{Re}(\lambda)>0} \|(\lambda I - \mathcal{L})^{-1}\|_{\mathcal{L}(G)} < \infty, \tag{8}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

we prove that system (5) is exponentially stable for $\beta \geq \alpha\tau$.

Proof of (7)

Here $\theta(0,t) = \theta(\ell,t) = 0$.
Let $\lambda \in \mathbb{C}_0$ and $F = (f,g,p,h) \in \mathcal{H}$. We look for $U = (u,v,z,\theta)$ such that

$$(\lambda I - \mathcal{A})U = F,$$

i.e.

$$\begin{cases} \lambda u - v = f, \quad \text{in} \quad H_0^1(\Omega), \\ \lambda v - (\alpha z(.,1) + \beta v_x)_x + \gamma\theta_x = g, \quad \text{in} \quad L^2(\Omega), \\ \lambda z + \frac{1}{\tau} z_\rho = p, \quad \text{in} \quad L^2(\Omega \times (0,1)), \\ \lambda\theta + \gamma v_x - \kappa\theta_{xx} = h, \quad \text{in} \quad L^2(\Omega). \end{cases} \tag{9}$$

We have,

$$z(x,\rho) = e^{-\lambda\tau\rho} u_x(x) + \tau e^{-\lambda\tau\rho} \int_0^\rho p(s) e^{\lambda\tau s} ds,$$

Proof of (7)

Multiplying $(9)_2$ and $(9)_4$ respectively by $\lambda w \in H_0^1(\Omega)$ and $\varphi \in H_0^1(\Omega)$, and summing:

$$B\left((u,\theta),(w,\varphi)\right) = \Phi(w,\varphi)$$

with

$$\begin{aligned} B\left((u,\theta),(w,\varphi)\right) &= \overline{\lambda}\lambda^2 \int_\Omega u\overline{w}dx + \left(\alpha\overline{\lambda}e^{-\lambda\tau} + |\lambda|^2\beta\right)\int_\Omega u_x\overline{w_x}dxdx \\ &+ \lambda\int_\Omega \theta\overline{\varphi} + \kappa\int_\Omega \theta_x\overline{\varphi_x}dx \\ &+ \gamma\left(\overline{\lambda}\int_\Omega \theta_x\overline{w}dx - \lambda\int_\Omega u\overline{\varphi_x}dx\right) \end{aligned}$$

$$\Phi((w,\varphi)) = \overline{\lambda}\int_\Omega (g+\lambda f)\overline{w}dx + \overline{\lambda}\int_\Omega(\beta f_x - \alpha z_0)\overline{w_x}dx + \int_\Omega (h+\gamma f_x)\overline{\theta}dx$$

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Lax-Milgram:

$$\mathcal{F} := \left\{(w,\varphi) \in H_0^1(\Omega)\times H_0^1(\Omega)\right\},$$

Proof of (7)

Coercivity of B:

$$\begin{aligned} Re\left(B\left((w,\varphi),(w,\varphi)\right)\right) &= Re(\lambda)|\lambda|^2\|w\|^2 + Re(\lambda)\|\varphi\|^2 + \kappa\|\varphi_x\|^2 \\ &+ \left(\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta\right)\|w_x\|^2. \end{aligned}$$

To conclude, it suffices to prove that $\left(\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta\right) > 0$.

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- $\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta \geq |\lambda|\left(|\lambda|\beta - \alpha\right) > 0$ for $|\lambda| \geq \frac{\alpha}{\beta}$.

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- $\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta \geq |\lambda|\left(|\lambda|\beta - \alpha\right) > 0$ for $|\lambda| \geq \frac{\alpha}{\beta}$.
- $|\lambda| \leq \frac{\alpha}{\beta}$. Denote $a := Re(\lambda)$ and $b = Im(\lambda)$:

$$\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta \geq \alpha e^{-a\tau} a\cos(b\tau) + \left(\beta - \alpha\tau e^{-a\tau}\right) b^2 + a^2\beta$$

By using $\alpha\tau \leq \beta$, we have $(\beta - \alpha\tau e^{-a\tau}) > 0$. Then

$$\alpha Re(\overline{\lambda}e^{-\lambda\tau}) + |\lambda|^2\beta > 0.$$

Proof of (8)

Suppose that condition (8) is false. Then there exists a sequence of complex numbers λ_n such that $Re(\lambda_n) > 0$, for all $n \in \mathbb{N}$, and $|\lambda_n| \to \infty$, and a sequence of vector $U_n = (u_n, v_n, z_n, \theta_n) \in \mathcal{D}(\mathcal{A})$ with $\|U_n\| = 1$ for every $n \in \mathbb{N}$, such that

$$\|(\lambda_n I - \mathcal{A})U_n\| = o(1). \tag{10}$$

i.e.

$$\begin{cases} \lambda u_n - v_n = f_n = o(1), \text{ in } H_0^1(\Omega), \\ \lambda_n v_n - (\alpha z_n(.,1) + \beta v_{n,x})_x + \gamma\theta_x = g_n = o(1), \text{ in } L^2(\Omega), \\ \lambda_n z_n + \frac{1}{\tau} z_{n,\rho} = p_n = o(1), \text{ in } L^2(\Omega \times (0,1)), \\ \lambda_n \theta_n + \gamma v_{n,x} - \kappa\theta_{n,xx} = h_n = o(1), \text{ in } L^2(\Omega). \end{cases} \tag{11}$$

We will prove that $\|U_n\| = o(1)$ which contradict the hypothesis: $\|U_n\| = 1$.

$$(\xi = \frac{2\tau\alpha^2}{\beta},\ m = \frac{\alpha^2}{\beta})$$

Proof of (8)

$$Re\left(\langle(\lambda_n I - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}}\right)$$

$$\geq Re(\lambda_n) + \frac{\beta}{2}\|v_{n,x}\|^2 - m\|u_{n,x}\|^2 + \kappa\|\theta_{n,x}\|^2$$

$$\geq Re(\lambda_n) + \frac{\beta}{2}\|v_{n,x}\|^2 - \frac{m}{|\lambda|^2}\|v_{n,x} + f_{n,x}\|^2 + \kappa\|\theta_{n,x}\|^2$$

Then

$$Re\left(\langle(\lambda_n I - \mathcal{A})U_n, U_n\rangle_{\mathcal{H}}\right)$$
$$\geq Re(\lambda_n) + \left(\tfrac{1}{2}\beta - \tfrac{2m}{|\lambda_n|^2}\right)\|v_{n,x}\|^2 + \kappa\|\theta_{n,x}\|^2 - \tfrac{2m}{|\lambda_n|^2}\|f_{n,x}\|^2$$

Hence

$$v_{n,x} = o(1) \quad \text{and} \quad \theta_{n,x} = o(1)$$

which yield, first, using again $(11)_1$,

$$u_{n,x} = o(1).$$

(K. Ammari, M. Salhi and F. Shel). H is a Hilbert space.
$A : D(A) \subset H \to H$ a self-adjoint, positive definite operator.

$$\begin{cases} u''(t) + Au(t-\tau) - A^{\beta}\theta(t) = 0, & t \in (0,+\infty), \\ \theta'(t) + A^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, & t \in (0,+\infty), \\ u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\ A^{1/2}u(t-\tau) = \phi(t), & t \in (0,\tau), \end{cases}$$

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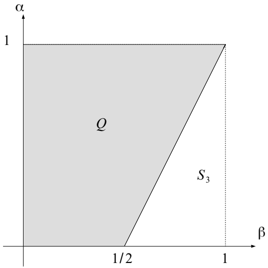
$$\begin{cases} u''(t) + Au(t-\tau) + aAu'(t) - A^{\beta}\theta(t) = 0, & t \in (0, +\infty), \\ \theta'(t) + A^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, & t \in (0, +\infty), \\ u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\ A^{1/2}u(t-\tau) = \phi(t-\tau), & t \in (0, \tau), \end{cases} \tag{12}$$

(K. Ammari, M. Salhi and F. Shel). H is a Hilbert space.
$A : D(A) \subset H \to H$ a self-adjoint, positive definite operator.

$$\begin{cases} u''(t)+Au(t-\tau)+aAu'(t)-A^{\beta}\theta(t)=0, & t\in(0,+\infty),\\ \theta'(t)+A^{\alpha}\theta(t)+A^{\beta}u'(t)=0, & t\in(0,+\infty),\\ u(0)=u_0, u'(0)=u_1, \theta(0)=\theta_0, & \\ A^{1/2}u(t-\tau)=\phi(t-\tau), & t\in(0,\tau), \end{cases} \tag{12}$$

$$Q := S\cup S_1\cup S_2=\{(\beta,\alpha)\in[0,1]\times[0,1] \mid 2\beta-\alpha\le 1\}.$$



$$z(\rho, t) := A^{1/2}x(t-\tau\rho), \quad \rho \in (0,1), t > 0.$$

Then, problem (12) is equivalent to

$$u''(t) + A^{1/2}z(1,t) + aAu'(t) - A^{\beta}\theta(t) = 0, \quad t > 0,$$

$$\theta'(t) + A^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, \quad t > 0,$$

$$\tau z_t(t,\rho) + z_\rho(\rho,t) = 0, \quad (\rho,t) \in (0,1) \times (0,+\infty),$$

$$u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0,$$

$$z(\rho, 0) = \phi(-\tau\rho), \quad \rho \in (0,1)$$

$$z(0,t) = A^{1/2}u(t), \quad t > 0.$$

Define $U = (u, u', \theta, z)^\top$, Then, problem (12) can be formulated as

$$\begin{cases} U' = \mathcal{A}_{\alpha,\beta} U, \\ U(0) = \left(u_0, u_1, \theta_0, \phi(-\tau.)\right)^\top, \end{cases} \tag{13}$$

$$\mathcal{A}_{\beta,\alpha} \begin{pmatrix} u \\ v \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} v \\ -A^{1/2}(z(.,1) + aA^{1/2}v - A^{\beta-\frac{1}{2}}\theta) \\ -A^{\alpha/2}\left(A^{\alpha/2}\theta + A^{\beta-\frac{\alpha}{2}}v\right) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix},$$

$$D(\mathcal{A}_{\beta,\alpha}) =$$

$$\left\{ \begin{array}{c} (u, v, \theta, z)^\top \in D(A^{1/2}) \times D(A^{1/2}) \times D(A^{\alpha/2}) \times H^1((0,1), H) : \\ z(0) = A^{1/2}u, \quad z(1) + aA^{1/2}v - A^{\beta-\frac{1}{2}}\theta \in D(A^{\frac{1}{2}}), \quad \text{and} \\ A^{\alpha/2}\theta + A^{\beta-\frac{\alpha}{2}}v \in D(A^{\alpha/2}) \end{array} \right\},$$

State space

$$\mathcal{H} = D(A^{1/2}) \times H \times H \times L^2\big((0,1), H\big),$$

equipped with the scalar product

$$\begin{aligned}\big((u, v, \theta, z), (u_1, v_1, \theta_1, z_1)\big)_{\mathcal{H}} &= \big(A^{1/2}u, A^{1/2}u_1\big)_H + (v, v_1)_H \\ &+ (\theta, \theta_1)_H + \xi \int_0^1 (z, z_1)_H d\rho.\end{aligned}$$

Theorem

For $(\beta, \alpha) \in Q$, $a \geq \tau$ and $\xi > \dfrac{2\tau}{a}$, the system (12) is well posed. More precisely, the operator $\mathcal{A}_{\beta,\alpha}$ generates a $\mathcal{C}_0$-semigroup on $\mathcal{H}$. Moreover, the $\mathcal{C}_0$-semigroup $e^{\mathcal{A}_{\beta,\alpha} t}$ is exponentially stable.

Application: Thermoelastic plate with delay

Taking $\alpha = \beta = \frac{1}{2}$, $H = L^2(\Omega)$ where Ω is a smooth open bounded domain in $\mathbb{R}^n$, and consider

$$\begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t-\tau) + a\Delta^2 u_t(x,t) + \Delta\theta(x,t) = 0, & (x,t) \in \Omega \times \\ \theta_t(x,t) - \Delta\theta(x,t) - \Delta u_t(x,t) = 0, & (x,t) \in \Omega \times \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \\ w(x,0) = w^0(x), w_t(x,0) = w^1(x), & t \in (0,\tau) \\ \theta(x,t) = 0, & (x,t) \in \partial\Omega \\ \theta(x,0) = \theta^0(x), & \\ -\Delta u(x,t) = f_0(x,t), & -\tau \le t < 0 \end{cases}$$

where τ and a are real positive constants.
Here, $A^{1/2} = -\Delta$, with domain $D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)$, and $A = -\Delta^2$, with domain $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$

$$
\begin{cases}
u''(t) + Au(t-\tau) + aAu'(t) - A^{\beta}\theta(t) = 0, & t \in (0, +\infty), \\
\theta'(t) + A^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, & t \in (0, +\infty), \\
u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\
B^* u(t-\tau) = \phi(t-\tau), & t \in (0, \tau),
\end{cases}
$$

$$\begin{cases} u''(t) + BB^*u(t-\tau) + aAu'(t) - C\theta(t) = 0, & t \in (0, +\infty), \\ \theta'(t) + A^\alpha\theta(t) + C^*u'(t) = 0, & t \in (0, +\infty), \\ u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\ B^*u(t-\tau) = \phi(t-\tau), & t \in (0, \tau), \end{cases} \tag{14}$$

$B : D(B) : H \to H$ and $C : D(C) : H \to H$ are closed densely defined linear operators.

Formally, system (14) can be seen as a generalization of the delayed α-β system (12).
We suppose also that $D(A^\alpha) \subset D(C)$ and $D(B^*) \subset D(C^*)$ and

$$\|C^*v\|_H \leq c_1\|B^*v\|_H, \ \ \forall\ v \in D(B^*)$$

.

Here we take

$$z(\rho, t) := B^* u(t - \tau\rho), \quad \rho \in (0,1), t > 0.$$

Then, problem (14) is equivalent to

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = \big(u_0, u_1, \theta_0, \phi(-\tau .)\big)^\top, \end{cases}$$

$$\mathcal{H} = D(B^*) \times H \times H \times L^2\big((0,1), H\big),$$

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$$\mathcal{H} = D(B^*) \times H \times H \times L^2\big((0,1), H\big),$$

System (14) is well-posed and exponentially stable for $a \geq \tau$.

Application: Thermoelastic string

$$\begin{cases} u_{tt}(t,x) - u_{xx}(x,t) - au_{xx}(x,t-\tau) + \theta_x(x,t) = 0, & (x,t) \in (0,L) \times \\ \theta_t(x,t) - \theta_{xx}(x,t) + u_{xt}(x,t) = 0, & (x,t) \in (0,L) \times \\ u(0,t) = u(L,t) = 0, & t \in (0,+\infty) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & t \in (0,\tau) \\ \theta(0,t) = \theta(L,t) = 0, & t \in (0,+\infty) \\ \theta(x,0) = \theta^0(x), & \\ u_x(x,t) = f_0(x,t), -\tau \leq t < 0, x \in \Omega, & \end{cases}$$

$H = L^2(0,L)$, $A = -\frac{\partial^2}{\partial x^2} : D(A) = H_0^1(0,L) \cap H^2(0,L) \to L^2(0,L)$,
$\alpha = 1$, $B = C^* = -\frac{\partial}{\partial x} : D(B) = H^1(0,L) \to L^2(0,L)$,
$B^* = C = \frac{\partial}{\partial x} : D(B^*) = H_0^1(0,L) \to L^2(0,L)$. We have that
$A = BB^*$, $D(A) \subset D(C)$, $D(B^*) \subset D(C^*)$ and
$\|C^*v\| \leq \|B^*v\|, \quad \forall\, v \in D(B^*)$.

$$\begin{cases} u''(t) + Au(t) - A^{\beta}\theta(t) = 0, & t \in (0, +\infty), \\ \theta'(t) + \kappa A^{\alpha}\theta(t-\tau) + aA^{\alpha}\theta(t) + A^{\beta}u'(t) = 0, & t \in (0, +\infty), \\ u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\ A^{\alpha/2}\theta(t-\tau) = g(t-\tau), & t \in (0, \tau). \end{cases}$$

where $\kappa > 0$ is a constant.

$$\begin{cases} u''(t) + Au(t) - C\theta(t) = 0, & t \in (0, +\infty), \\ \theta'(t) + \kappa BB^*\theta(t-\tau) + aBB^*\theta(t) + B^*u'(t) = 0, & t \in (0, +\infty), \\ u(0) = u_0, u'(0) = u_1, \theta(0) = \theta_0, & \\ B^*\theta(t-\tau) = \phi(t), & t \in (0, \tau), \end{cases} \tag{15}$$

We suppose that $D(A^{1/2}) \subset D(C^*)$ and $D(B^*) \subset D(C)$ and

$$\|Cv\|_H \leq c\|B^*v\|_H, \;\; \forall \; v \in D(B^*)$$

Application

$$\begin{cases} w_{tt}(t,x) + w_{xx}(x,t) + \theta_x(x,t) = 0, & (x,t) \in (0,L) \\ \theta_t(x,t) + \kappa\theta_{xx}(x,t-\tau) + a\theta_{xx}(x,t) + u_{xt}(x,t) = 0, & (x,t) \in (0,L) \\ w(0,t) = w(L,t) = 0, & \\ w(x,0) = w^0(x), w_t(x,0) = w^1(x), & t \in (0,\tau) \\ \theta_x(0,t) = \theta_x(L,t) = 0, & t \in (0,+\infty) \\ \theta(x,0) = \theta^0(x), & \\ w_x(x,t) = f_0(x,t), & -\tau \le t < 0, x \end{cases}$$

THANKS!