Wave equations with nonsmooth boundary conditions: well-posedness and asymptotic behavior

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Control Theory & Inverse Problems

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Based on the paper

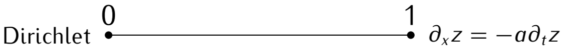
Y. Chitour, S. Marx, G. Mazanti.
One-dimensional wave equation with set-valued boundary damping: well-posedness, asymptotic stability, and decay rates.
ESAIM Control Optim. Calc. Var., 27:Paper No. 84, 62 pp., 2021.
https://doi.org/10.1051/cocv/2021067

Outline

Introduction

Linear wave equation with linear boundary conditions

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) & (t,x) \in \mathbb{R}_+ \times (0,1) \\ z(t,0) = 0 & t \in \mathbb{R}_+ \\ \partial_x z(t,1) = -a\partial_t z(t,1) & t \in \mathbb{R}_+ \end{cases} \qquad a \in \mathbb{R}$$



Introduction

Linear wave equation with linear boundary conditions

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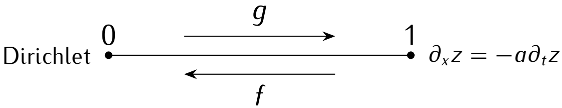


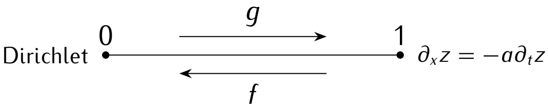
D'Alembert decomposition intro traveling waves:

$$z(t,x) = \frac{1}{\sqrt{2}} \int_0^{t+x} f(s)\,\mathrm{d}s + \frac{1}{\sqrt{2}} \int_0^{t-x} g(s)\,\mathrm{d}s$$

$f : [0,+\infty) \to \mathbb{R}$, $g : [-1,+\infty) \to \mathbb{R}$: Riemann invariants

Introduction

Linear wave equation with linear boundary conditions



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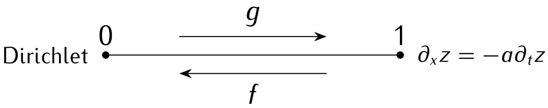
Linear wave equation with linear boundary conditions



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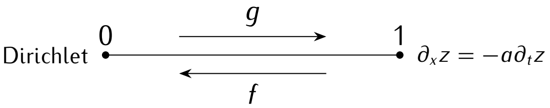
- At $x = 0$:

$$z(t, 0) = 0 \implies \partial_t z(t, 0) = 0 \implies g(t) = -f(t)$$

Introduction

Linear wave equation with linear boundary conditions



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$$\partial_t z(t,x) = \frac{f(t+x)+g(t-x)}{\sqrt{2}}$$

$$\partial_x z(t,x) = \frac{f(t+x)-g(t-x)}{\sqrt{2}}$$

- At $x = 0$:
  $z(t,0) = 0 \implies \partial_t z(t,0) = 0 \implies g(t) = -f(t)$
- At $x = 1$:
  $\partial_x z(t,1) = -a\partial_t z(t,1) \implies f(t+1) = \frac{1-a}{1+a} g(t-1)$
  $\Gamma = \frac{1-a}{1+a}$: reflection coefficient

$$\boxed{g(t) = -\Gamma g(t-2)}$$

Introduction

Linear wave equation with linear boundary conditions

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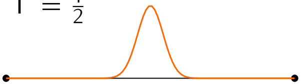
$$g(t) = -\Gamma g(t-2) \qquad \Gamma = \frac{1-a}{1+a}$$

- Existence and uniqueness: ok if $a \neq -1$
- $\begin{cases} \partial_t z(t+2,x) = -\Gamma \partial_t z(t,x) \\ \partial_x z(t+2,x) = -\Gamma \partial_x z(t,x) \end{cases}$

$\Gamma = \frac{1}{2}$

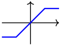


- $\begin{cases} a > 0,\ a \neq 1 & 0 < \lvert\Gamma\rvert < 1 & \text{Exponential convergence} \\ a = 1 & \Gamma = 0 & \text{Finite-time convergence} \\ a < 0,\ a \neq -1 & \lvert\Gamma\rvert > 1 & \text{Exponential instability} \\ a = -1 & & \text{Not well-posed} \\ a = 0 \text{ or } a \to \infty & \lvert\Gamma\rvert = 1 & \text{Periodic solutions} \end{cases}$

Introduction

Nonlinear boundary condition

$$\begin{cases} \partial^2_{tt} z(t,x) = \partial^2_{xx} z(t,x) & (t,x) \in \mathbb{R}_+ \times (0,1) \\ z(t,0) = 0 & t \in \mathbb{R}_+ \\ \partial_x z(t,1) = -\sigma(\partial_t z(t,1)) & t \in \mathbb{R}_+ \end{cases} \qquad \sigma : \mathbb{R} \to \mathbb{R}$$

- Motivation: Localized boundary control
  $\partial_x z(t,1) = u(t) \qquad u(t) = -\sigma(\partial_t z(t,1))$
- Nonlinear phenomena in the implementation of a control strategy: nonlinearities in components, saturation: 
- Subject of several works, also in higher dimension: [Conrad, Leblond, Marmorat; 1989], [Zuazua; 1990], [Lasiecka, Tataru; 1993], [Martinez; 1999], [Pierre, Vanconstenoble; 2000], [Vancostenoble, Martinez; 2000], [Haraux; 2009], [Alabau-Boussouira; 2012], [Cheng-Zhong Xu, Gen Qi Xu; 2019], ...

Introduction

Nonlinear boundary condition

Main questions

(Q1) Existence and uniqueness of solutions

(Q2) Asymptotic behavior as $t \to +\infty$

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ \partial_x z(t,1) = -\sigma(\partial_t z(t,1)) \end{cases}$$

- Classical functional framework: $(z(t,\cdot), \partial_t z(t,\cdot)) \in X_2 := H^1_*(0,1) \times L^2(0,1)$ with $H^1_*(0,1) = \{z \in H^1(0,1) \mid z(0) = 0\}$
  - ⇝ Some works consider L^p in dimension $d = 1$
  - ⇝ Ill-posed in L^p for $p \neq 2$ and $d \geq 2$ [Peral; 1980]
- σ is usually assumed to be continuous, nondecreasing, and with $\sigma(0) = 0$
- Energy is nonincreasing $\iff$ $\sigma(s)s \geq 0 \ \forall s \in \mathbb{R}$

Introduction

Nonlinear boundary condition

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$$z(t,x) = \frac{1}{\sqrt{2}} \int_0^{t+x} f(s)\,\mathrm{d}s + \frac{1}{\sqrt{2}} \int_0^{t-x} g(s)\,\mathrm{d}s$$

$$\partial_t z(t,x) = \frac{f(t+x)+g(t-x)}{\sqrt{2}} \qquad \partial_x z(t,x) = \frac{f(t+x)-g(t-x)}{\sqrt{2}}$$

At $x = 0$: $z(t,0) = 0 \implies \partial_t z(t,0) = 0 \implies g(t) = -f(t)$

At $x = 1$: $\partial_x z(t,1) = -\sigma(\partial_t z(t,1))$

$$\implies \frac{f(t+1)-g(t-1)}{\sqrt{2}} = -\sigma\left(\frac{f(t+1)+g(t-1)}{\sqrt{2}}\right)$$

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$$\implies \frac{g(t)+g(t-2)}{\sqrt{2}} = \sigma\left(\frac{g(t-2)-g(t)}{\sqrt{2}}\right) \quad \text{since } f = -g$$

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$$\implies \sigma\left(\frac{g(t-2)-g(t)}{\sqrt{2}}\right) + \frac{g(t-2)-g(t)}{\sqrt{2}} = \sqrt{2}g(t-2)$$

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$$\implies \frac{g(t-2)-g(t)}{\sqrt{2}} = (\mathrm{id}+\sigma)^{-1}\left(\sqrt{2}g(t-2)\right) \quad \text{if id}+\sigma \text{ invert.}$$

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$$\implies \sigma\left(\frac{g(t-2)-g(t)}{\sqrt{2}}\right) + \frac{g(t-2)-g(t)}{\sqrt{2}} = \sqrt{2}g(t-2)$$

$$\implies \frac{g(t-2)-g(t)}{\sqrt{2}} = (\mathrm{id}+\sigma)^{-1}\left(\sqrt{2}g(t-2)\right) \quad \text{if } \mathrm{id}+\sigma \text{ invert.}$$

$$\implies g(t) = g(t-2) - \sqrt{2}(\mathrm{id}+\sigma)^{-1}\left(\sqrt{2}g(t-2)\right) =: S(g(t-2))$$

Introduction

Nonlinear boundary condition

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$$g(t) = S(g(t-2)) := g(t-2) - \sqrt{2}(\mathrm{id} + \sigma)^{-1}\left(\sqrt{2} g(t-2)\right)$$

- Existence and uniqueness if $\mathrm{id} + \sigma$ is invertible
- Previous argument adapted from [Pierre, Vanconstenoble; 2000]
  - ⇝ Used there to prove existence even if $\mathrm{id} + \sigma$ is not invertible (through a pseudo-inverse)
  - ⇝ Uniqueness condition: $\frac{\sigma(s)-\sigma(t)}{s-t} > -1$ if $s \neq t$
- What lies behind the above formula?

Introduction

Rotation of angle $-\pi/4$

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What lies behind the above formula?

$$\begin{cases} \partial_{tt}^2 z(t, x) = \partial_{xx}^2 z(t, x) & (t, x) \in \mathbb{R}_+ \times (0, 1) \\ z(t, 0) = 0 & t \in \mathbb{R}_+ \\ \partial_x z(t, 1) = -\sigma(\partial_t z(t, 1)) & t \in \mathbb{R}_+, \quad \sigma\colon \mathbb{R} \to \mathbb{R} \end{cases}$$

Introduction

Rotation of angle $-\pi/4$

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Σ: graph of σ

Introduction

Rotation of angle $-\pi/4$

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$$\begin{pmatrix} g(t-x) \\ -f(t+x) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R} \begin{pmatrix} \partial_t z(t,x) \\ -\partial_x z(t,x) \end{pmatrix}$$

Boundary conditions in terms of f and g:

$$\begin{aligned} z(t,0) = 0 \quad &\iff \quad g(t) = -f(t) \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \quad &\iff \quad (g(t-1), -f(t+1)) \in R\Sigma \end{aligned}$$

In terms of g only: $\boxed{(g(t-2), g(t)) \in R\Sigma}$ $\quad \forall t \geq 1$

Introduction

Rotation of angle $-\pi/4$

$$g(t) = S(g(t-2)) := g(t-2) - \sqrt{2}(\mathrm{id} + \sigma)^{-1}\left(\sqrt{2}g(t-2)\right)$$

What lies behind the above formula?

$$\boxed{(g(t-2), g(t)) \in R\Sigma} \qquad \forall t \geq 1$$

$$\Sigma\colon \text{graph of } \sigma, \qquad R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

S is (the function whose graph is) the rotation of the graph of σ by an angle $-\frac{\pi}{4}$

Wave equation with set-valued boundary damping

Setting

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) & (t,x) \in \mathbb{R}_+ \times (0,1) \\ z(t,0) = 0 & t \in \mathbb{R}_+ \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma & t \in \mathbb{R}_+ \end{cases} \qquad \Sigma \subset \mathbb{R}^2$$

- Previous setting: Σ is the graph of $\sigma\colon \mathbb{R} \to \mathbb{R}$
- Now: Σ is the graph of a set-valued function $\mathbb{R} \rightrightarrows \mathbb{R}$

$$z(t,x) = \frac{1}{\sqrt{2}} \int_0^{t+x} f(s)\,\mathrm{d}s + \frac{1}{\sqrt{2}} \int_0^{t-x} g(s)\,\mathrm{d}s$$

By the same D'Alembert decomposition as before, the above system is equivalent to

$$g(t) \in S(g(t-2)) \qquad \forall t \geq 1$$

where $S\colon \mathbb{R} \rightrightarrows \mathbb{R}$ the set-valued map whose graph is $R\Sigma$.

Wave equation with set-valued boundary damping

Functional setting

Solution z	Function g
$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \end{cases}$	$g(t) \in S(g(t-2))$
$(z(t,\cdot), \partial_t z(t,\cdot)) \in \underbrace{W_*^{1,p}(0,1) \times L^p(0,1)}_{X_p}$	$g \in L^p_{\mathrm{loc}}(-1,+\infty)$

$W_*^{1,p}(0,1) = \{z \in W^{1,p}(0,1) \mid z(0) = 0\}$ Graph$(S) = R\Sigma$

$$\|(u,v)\|_{X_p} = \begin{cases} \frac{1}{\sqrt{2}} \left[\int_0^1 \left(|u' + v|^p + |u' - v|^p\right) \mathrm{d}s\right]^{\frac{1}{p}} & p < +\infty \\ \frac{1}{\sqrt{2}} \max\left(\|u' + v\|_{L^\infty}, \|u' - v\|_{L^\infty}\right) & p = +\infty \end{cases}$$

$$\|(z(t,\cdot), \partial_t z(t,\cdot))\|_{X_p} = \|g(t+\cdot)\|_{L^p(-1,1)}$$

Wave equation with set-valued boundary damping

Functional setting

Solution z	Function g
$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \end{cases}$	$g(t) \in S(g(t-2))$
$(z(t,\cdot), \partial_t z(t,\cdot)) \in \underbrace{W_*^{1,p}(0,1) \times L^p(0,1)}_{\mathrm{X}_p}$	$g \in L^p_{\mathrm{loc}}(-1,+\infty)$

$$W_*^{1,p}(0,1) = \{z \in W^{1,p}(0,1) \mid z(0) = 0\}$$

$\mathrm{Graph}(S) = R\Sigma$

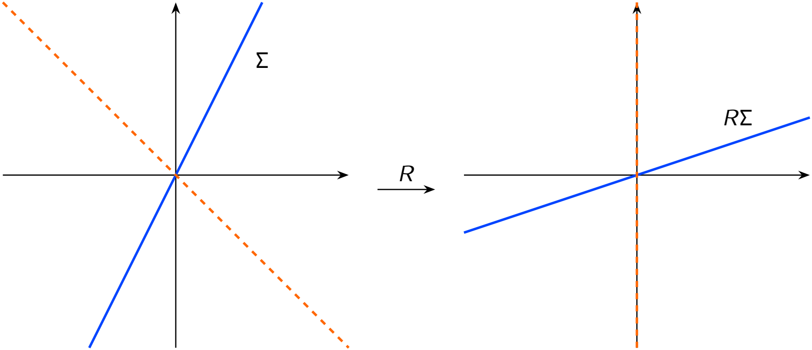
$$\|(u,v)\|_{\mathrm{X}_p} = \begin{cases} \dfrac{1}{\sqrt{2}} \left[\displaystyle\int_0^1 \left(|u' + v|^p + |u' - v|^p\right) \mathrm{d}s\right]^{\frac{1}{p}} & p < +\infty \\ \dfrac{1}{\sqrt{2}} \max\left(\|u' + v\|_{L^\infty}, \|u' - v\|_{L^\infty}\right) & p = +\infty \end{cases}$$

$$\sout{\|(z(t,\cdot), \partial_t z(t,\cdot))\|_{\mathrm{X}_p}}\ \|z(t,\cdot)\|_{\mathrm{X}_p} = \|g(t+\cdot)\|_{L^p(-1,1)}$$

Wave equation with set-valued boundary damping

The set-valued map S

$S\colon \mathbb{R} \rightrightarrows \mathbb{R}$ is the set-valued map whose graph is $R\Sigma$



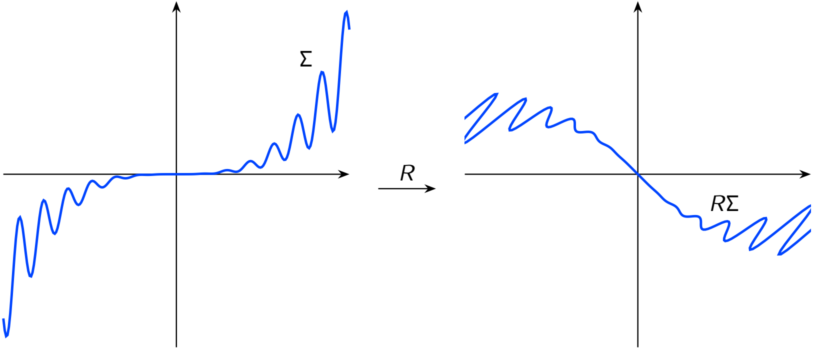
Σ: graph of $\sigma(x) = ax$, $R\Sigma$: graph of $x \mapsto -\Gamma x$, $\Gamma = \frac{1-a}{1+a}$

If $a = -1$, then $R\Sigma$ is a vertical line

Wave equation with set-valued boundary damping

The set-valued map S

$S\colon \mathbb{R} \rightrightarrows \mathbb{R}$ is the set-valued map whose graph is $R\Sigma$



Σ is the graph of a function σ

Wave equation with set-valued boundary damping

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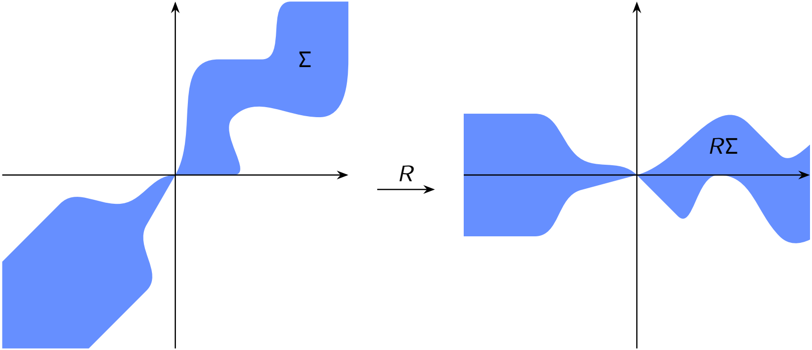

Σ and $R\Sigma$ are not graphs of functions

Wave equation with set-valued boundary damping

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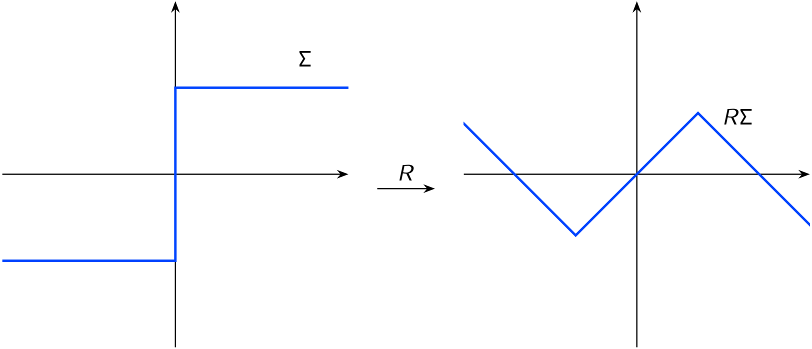


Σ is the sign set-valued map, $R\Sigma$ is the graph of a function

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \end{cases}$$

$$g_{n+1}(t) \in S(g_n(t))$$
$$\mathrm{Graph}(S) = R\Sigma$$

Theorem (Existence)

Assume that $R\Sigma$ contains the graph of a universally measurable function with linear growth. Then, for every initial condition $(z(0,\cdot), \partial_t z(0,\cdot))$ in X_p, there exists a solution to the wave equation such that $(z(t,\cdot), \partial_t z(t,\cdot)) \in X_p$ for all $t \geq 0$.

- φ is universally measurable $\iff$ $\varphi \circ g$ is Lebesgue meas. $\forall g$ Lebesgue meas.
- φ with linear growth: $g \in L^p \implies \varphi \circ g \in L^p$
- "Conversely", if $\exists$ a solution for every initial condition in X_p, then $R\Sigma$ contains the graph of a universally measurable function and the graph of a function with linear growth

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

$$\begin{cases} \partial_{tt}^2 z(t, x) = \partial_{xx}^2 z(t, x) \\ z(t, 0) = 0 \\ (\partial_t z(t, 1), -\partial_x z(t, 1)) \in \Sigma \end{cases}$$

$$g_{n+1}(t) \in S(g_n(t))$$
$$\mathrm{Graph}(S) = R\Sigma$$

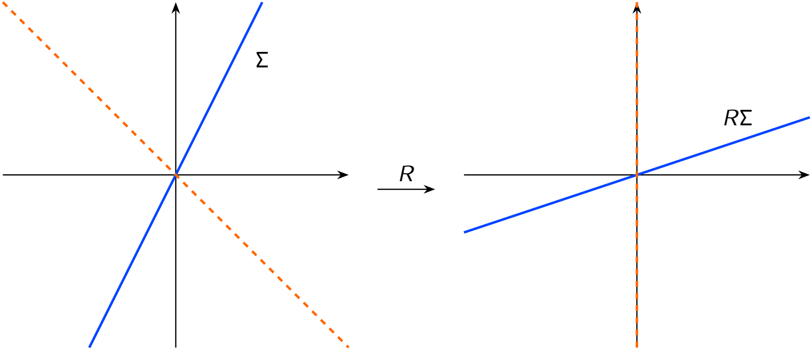
Theorem (Uniqueness)

For every initial condition in X_p, there exists a unique solution to the wave equation if and only if $R\Sigma$ is equal to the graph of a universally measurable function with linear growth.

- Necessary and sufficient condition in terms of $R\Sigma = \mathrm{Graph}(S)$
- Both statements only for $p < +\infty$; also hold for $p = +\infty$ replacing linear growth by a weaker assumption

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

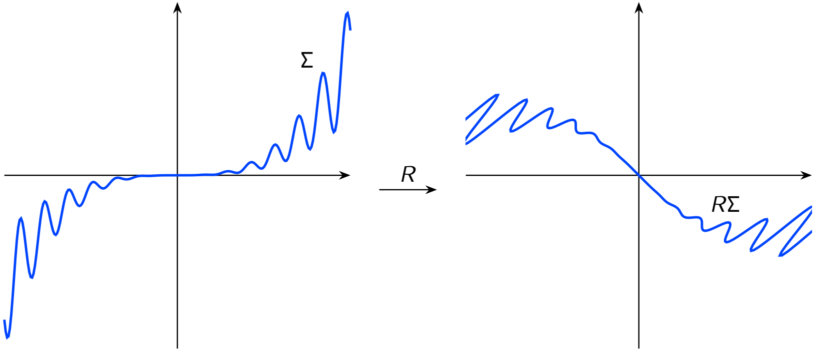


Σ and $R\Sigma$ are graphs of linear functions

Existence and uniqueness except if Σ is the graph of $\sigma(x) = -x$

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

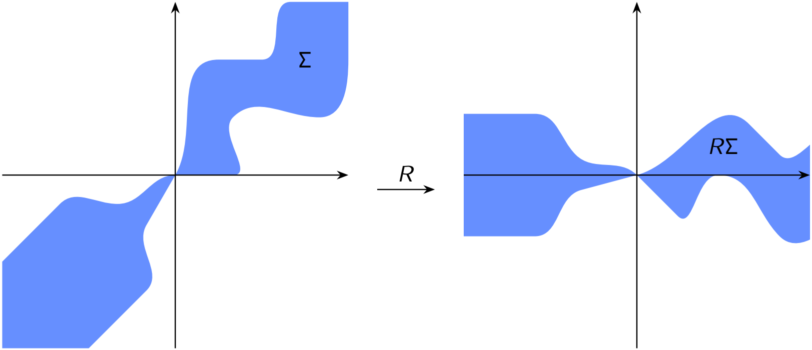


Σ is the graph of a function σ

Existence, not uniqueness

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

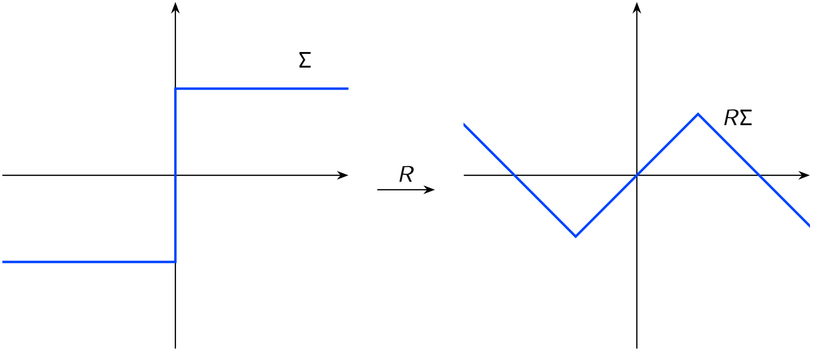


Σ and $R\Sigma$ are not graphs of functions

Existence, not uniqueness

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions

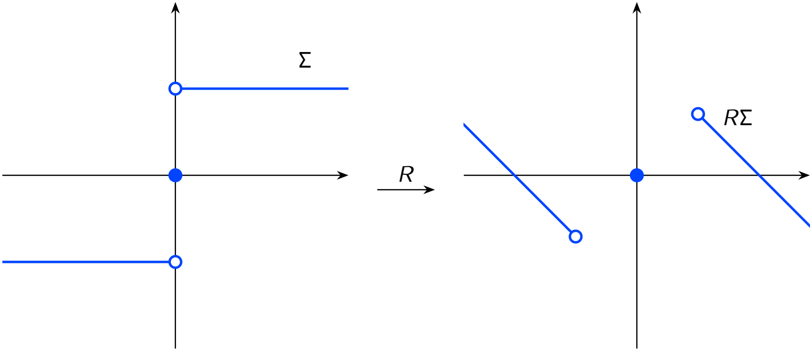


Σ is the sign set-valued map, $R\Sigma$ is the graph of a function

Existence and uniqueness

Wave equation with set-valued boundary damping

Existence and uniqueness of solutions



Σ is the sign function

No existence

Asymptotic behavior

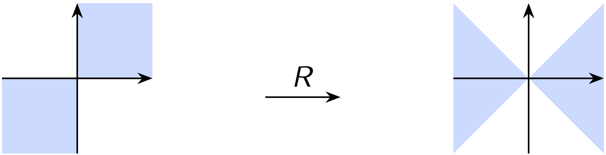
Nonincreasing norm

From now on: Σ is such that we have existence of solutions (not necessarily uniqueness)

Proposition

For every solution, $t \mapsto \|z(t, \cdot)\|_{X_p}$ is nonincreasing
$\iff \forall (x, y) \in \Sigma,\ xy \geq 0$
$\iff \forall (x, y) \in R\Sigma,\ |y| \leq |x|$

Generalization of the condition $s\sigma(s) \geq 0$



We always assume this condition from now on

Asymptotic behavior

Some previous results

Some previous results with nonlinear damping

[Vancostenoble, Martinez; 2000]

[Alabau-Boussouira; 2012]

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ \partial_x z(t,1) = -\sigma(\partial_t z(t,1)) \end{cases}$$

(σ is odd, expressions below for $s > 0$ small and t large)

$$\sigma(s) = s^q \implies \|z(t)\|_{X_2} \sim t^{-\frac{1}{q-1}} \qquad q > 1$$

$$\sigma(s) = s^q \left(\ln\left(\frac{1}{s}\right)\right)^r \implies \|z(t)\|_{X_2} \sim t^{-\frac{1}{q-1}} (\ln t)^{-\frac{r}{q-1}} \qquad q > 1, r > 0$$

$$\sigma(s) = e^{-\frac{1}{s^q}} \implies \|z(t)\|_{X_2} \sim (\ln t)^{-\frac{1}{q}} \qquad q > 0$$

$$\sigma(s) = e^{-e^{1/s}} \implies \|z(t)\|_{X_2} \sim (\ln \ln t)^{-2}$$

$$\sigma(s) = e^{-\left(\ln\left(\frac{1}{s}\right)\right)^q} \implies \|z(t)\|_{X_2} \sim e^{-(\ln t)^{\frac{1}{q}}} \qquad 1 < q < 2$$

$$\sigma(s) = s \left(\ln\left(\frac{1}{s}\right)\right)^{-q} \implies \|z(t)\|_{X_2} \lesssim e^{-C t^{\frac{1}{q+1}}} \qquad q > 0$$

- $\sigma'(0) = 0$ in all of the above cases
- $\sigma \rightsquigarrow \sigma^{-1} \implies$ Same convergence rate

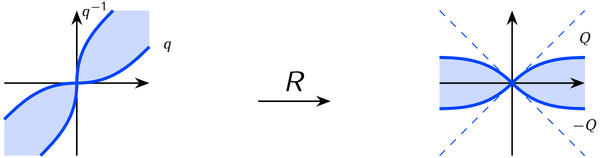
Asymptotic behavior

Decay rates

How fast do solutions converge to 0?

Nonlinear sector condition:

- In terms of Σ: $\exists q \in \mathcal{C}^1$ such that $q(0) = 0$, $0 < q(x) < x$, $|q'(x)| < 1$ for $x > 0$ such that
$$q(|x|) \leq |y| \quad \text{and} \quad q(|y|) \leq |x| \qquad \forall (x, y) \in \Sigma$$
- In terms of S: $\exists Q \in \mathcal{C}^1$ and $M > 0$ such that $Q(0) = 0$, $0 < Q(x) < x$, $Q'(x) > 0$ for $x > 0$ such that
$$|y| \leq Q(|x|) \qquad \forall y \in S(x)$$



$$Q(x) = \sqrt{2}(q + \mathrm{id})^{-1}(\sqrt{2}x) - x$$

Asymptotic behavior

Decay rates

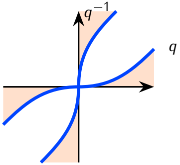
Theorem

Assume that Σ satisfies a nonlinear sector condition with functions q and Q as before. Then

$$\|z(t,\cdot)\|_{X_\infty} \leq Q^{[\lfloor \frac{t}{2} \rfloor]}(\|z(0,\cdot)\|_{X_\infty})$$

$$Q^{[n]} = \underbrace{Q \circ \cdots \circ Q}_{n}$$

- Similar statement for $p < +\infty$ but with additional terms
- $\|z(t,\cdot)\|_{X_p} \geq C_1 Q^{[\lfloor \frac{t}{2} \rfloor]}(C_2)$ for non-trivial solutions if we assume:



- If Σ is the graph of q or q^{-1}: $\|z(t,\cdot)\|_{X_p} \sim Q^{[\lfloor \frac{t}{2} \rfloor]}(C)$

Asymptotic behavior

Decay rates

Asymp. behavior of $\|(z(t,\cdot),\partial_t z(t,\cdot))\|_{X_p}$ $\leftrightsquigarrow$ Asymp. behavior of $Q^{[n]}(x_0)$

Asymptotic behavior of $Q^{[n]}(x_0)$ under nonlinear sector condition:

1. Case $q'(0) = 0$:
$$Q^{[n]}(x_0) \sim V(n)$$
where $V'(t) = -\sqrt{2}q(\sqrt{2}V(t))$ with $V(0) = x_0$
$\rightsquigarrow$ [Vancostenoble, Martinez; 2000]: $Q^{[n]}(x_0) = V(t_n)$ with $t_n \sim n$
2. Case $q'(0) \in (0,1)$: $\exists C > 1$ s.t.
$$C^{-1}e^{-\lambda n} \leq Q^{[n]}(x_0) \leq Ce^{-\lambda n}$$
where $\lambda = 2\operatorname{artanh}(q'(0))$.
3. Case $q'(0) = 1$:
$$\lim_{n\to+\infty} e^{\lambda n}Q^{[n]}(x_0) = 0 \qquad \text{for every } \lambda > 0$$

Remark: some results require additional technical assumptions, but weaker conclusions available also without those assumptions

Asymptotic behavior

Decay rates

Consequences:

(σ is odd, expressions below for $s > 0$ small and t large)

$$\sigma(s) = s^q \implies \|z(t)\|_{X_p} \sim t^{-\frac{1}{q-1}} \qquad q > 1$$

$$\sigma(s) = s^q \left(\ln\left(\tfrac{1}{s}\right)\right)^r \implies \|z(t)\|_{X_p} \sim t^{-\frac{1}{q-1}} (\ln t)^{-\frac{r}{q-1}} \qquad q > 1, r > 0$$

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$$\sigma(s) = e^{-\left(\ln\left(\frac{1}{s}\right)\right)^q} \implies \|z(t)\|_{X_p} \sim e^{-(\ln t)^{\frac{1}{q}}} \qquad 1 < q < 2$$

$$\sigma(s) = s \left(\ln\left(\tfrac{1}{s}\right)\right)^{-q} \implies \|z(t)\|_{X_p} \sim \frac{1}{\sqrt{2}} e^{-\sum_{k=0}^{N} \alpha_k t^{\frac{1-2qk}{q+1}}} \qquad q > 0$$

$$N = \left\lfloor \frac{1}{2q} \right\rfloor, \ \alpha_0 = (q+1)^{\frac{1}{q+1}}$$

Further comments

Other results

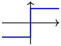
Also in [Chitour, Marx, Mazanti; 2021]:

- Necessary and sufficient condition for strong stability, uniform global asymptotic stability, and global exponential stability
- Arbitrarily slow convergence if σ is saturated: $\forall p \in [1, +\infty)$ $\forall \varphi\colon [0, +\infty) \to (0, +\infty)$ decreasing to 0, $\exists$ an initial condition in X_p s.t. $\forall$ solution z
$$0 < \varphi(t) \leq \|z(t, \cdot)\|_{X_p} \xrightarrow[t \to +\infty]{} 0$$
  - ⇝ Conjectured in [Vancostenoble, Martinez; 2000]
  - ⇝ Initial conditions with explosions (whence $p < +\infty$)

Further comments

Other results

Also in [Chitour, Marx, Mazanti; 2021]:

- Study of the case $\Sigma =$ 
  Solutions converge to a 2-periodic solution (in finite time if the initial condition is in X_∞)
  - ⇝ Extends a previous result of [Cheng-Zhong Xu, Gen Qi Xu; 2019] in $p = 2$ to the case of any $p \in [1, +\infty]$
  - ⇝ The limit is more explicitly identified (instead of based on a Fourier series expansion)
- Input-to-state stability (ISS) for systems with a boundary disturbance

$$(\partial_t z(t, 1), -\partial_x z(t, 1)) \in \Sigma + d(t)$$

Further comments

Ongoing work

Ongoing work with Y. Chitour and S. Marx: Generalize the approach to hyperbolic systems of the form

$$\begin{cases} \partial_t \mathbf{u}(t,x) + \Lambda(t,x)\partial_x \mathbf{u}(t,x) = \mathbf{g}(t,x,\mathbf{u}(t,\cdot)) & t > 0,\ x \in (0,1) \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) & x \in [0,1] \\ \mathbf{u}(\cdot,0) \in F(\mathbf{u}(\cdot,1)) & t \geq 0 \end{cases}$$

with $\mathbf{u}(t,x) \in \mathbb{R}^d$, $\Lambda(t,x)$ diagonal with positive diagonal entries, and F set-valued.

- Existence of solutions in $L^p_{t,x}$
- Hidden regularity: $\mathbf{u} \in \mathcal{C}^0_t L^p_x$ and $u \in \mathcal{C}^0_x L^p_t$
- Energy estimate:
  $\|\mathbf{u}\|_{L^p_{t,x};\, \mathcal{C}^0_t L^p_x;\, \mathcal{C}^0_x L^p_t} \lesssim \|\mathbf{u}_0\|_{L^p_x} + \|\mathbf{g}(\cdot,\cdot,\mathbf{0})\|_{L^p_{t,x}} + B$
  - ⇝ B depends on the behavior of F close to 0
- Uniqueness if F is single-valued
- Asymptotic behavior?