CTIP 2023-Control Theory and Inverse Problems

Sliding mode control for a class of linear infinite-dimensional systems

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Introduction

Context: Sliding mode control design for the stabilization of a class of infinite-dimensional systems.
Linear abstract Cauchy problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z = Az + B(u + d), \\ z(0) = z_0, \end{cases} \tag{1}$$

1. $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$,
2. H denotes a Hilbert space over the field $\mathbb{K}$,
3. $A : D(A) \subseteq H \to H$ is a linear operator with $D(A)$ densely defined in H,
4. $B \in \mathcal{L}(\mathbb{K}, D(A^*)')$, with A^* the adjoint operator of A,
5. $z(t) \in H$ is the state, $u(t) \in \mathbb{K}$ is the control input and $d(t) \in \mathbb{K}$ is an unknown disturbance.

Introduction

Example

$$
\begin{cases}
z_t(t,x) = z_{xx}(t,x), \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\
z_x(t,0) = c_0 z(t,0), \quad t \in \mathbb{R}_+, \\
z_x(t,1) = u(t) + d(t), \quad t \in \mathbb{R}_+, \\
z(0,x) = z_0(x),
\end{cases} \tag{2}
$$

$$
\begin{cases}
z_{tt}(t,x) + z_{xxxx}(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\
z(t,0) = z_x(t,0) = 0, \quad t \in \mathbb{R}_+, \\
z_{xx}(t,1) = 0, \quad t \in \mathbb{R}_+, \\
z_{xxx}(t,1) = u(t) + d(t), \quad t \in \mathbb{R}_+, \\
z(0,x) = z_0(x),
\end{cases} \tag{3}
$$

Introduction

Question

How can one propose a systematic methodology for the design of sliding variables for linear infinite-dimensional systems?

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Finite-dimensional example

Let us consider the following system

$$\begin{cases} \dot{z}_1 = -z_1 + z_2, \\ \dot{z}_2 = u + d. \end{cases} \quad (4)$$

Nominal control

If $d = 0$, the feedback-law

$$u := u_0 = -2z_2 \quad (5)$$

provides asymptotic stability of the origin of (4).

Finite-dimensional example

If we select u as

$$u = u_0 + u_{SMC} \tag{6}$$

then system (4) can be written as follows:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = A_L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + B(u_{SMC} + d) \tag{7}$$

with

$$A_L = A + BL,\ A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \text{ and } L = \begin{pmatrix} 0 & -2 \end{pmatrix}. \tag{8}$$

Finite-dimensional example

Let $\varphi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ be an eigenvector of $A_L^\top$ and let us introduce the following surface

$$\Sigma := \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^2 \mid \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^2} = z_1 + z_2 = 0 \right\}. \tag{9}$$

On Σ the system (7) is equivalent to

$$\begin{cases} \dot{z}_1 = -z_1 + z_2, \\ z_2 = -z_1. \end{cases} \tag{10}$$

Thus, z_1 and z_2 converge to zero asymptotically.

Finite-dimensional example

Let us introduce a new variable σ given by

$$\sigma = \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^2}. \tag{11}$$

From (7), the σ-dynamics yields

$$\begin{aligned} \dot{\sigma} &= \left\langle \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^2} = \left\langle A_L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + B(u_{SMC} + d), \varphi \right\rangle_{\mathbb{R}^2} \\ &= \left\langle A_L \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \varphi \right\rangle_{\mathbb{R}^2} + \langle B, \varphi \rangle_{\mathbb{R}^2} (u_{SMC} + d) \end{aligned} \tag{12}$$

Since φ is an eigenvector of A_L^{T} and $\langle B, \varphi \rangle_{\mathbb{R}^2} = 1$, then we obtain

$$\dot{\sigma} = \lambda \sigma + u_{SMC} + d. \tag{13}$$

Finite-dimensional example

Thus, the following holds, for all $t \geq 0$

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 \quad = \sigma\dot{\sigma} = \sigma\left(\lambda\sigma + u_{SMC} + d\right) = \lambda\sigma^2 + \sigma\left(u_{SMC} + d\right). \tag{14}$$

Since $\lambda = -1 < 0$ then, we have

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 \leq +\sigma u_{SMC} + |\sigma||d|. \tag{15}$$

Therefore, if we assume that d is bounded, i.e $\|d\|_{L^\infty(\mathbb{R}_+)} \leq M$ with $M > 0$, then by selecting

$$u_{SMC} = -\rho\,\mathrm{sign}(\sigma) \tag{16}$$

with $\rho > M$, we obtain

$$\frac{1}{2}\frac{d}{dt}|\sigma|^2 \leq -|\sigma|(\rho - M). \tag{17}$$

Finite-dimensional example

As a consequence, σ reaches zero in a finite-time t_r that is bounded by

$$t_r \leq \frac{|\sigma(0)|}{\rho - M}. \tag{18}$$

Conclusion

With the control input $u = -2z_2 - \rho \mathrm{sign}(\sigma)$, the system (4) reaches the sliding surface Σ in finite time t_r and remains on it.

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Assumption 1

The following statements hold.

(i) The operator $A : D(A) \subseteq H \to H$ generates a strongly continuous semigroup, that is denoted by $(\mathbb{T}(t))_{t \geq 0}$.

(ii) The operator B is admissible for $(\mathbb{T}(t))_{t \geq 0}$.

Assumption 1

(iii) There exists an operator $L : D(L) \to \mathbb{K}$ such that the operator

$$\begin{cases} A_L = A + BL, \\ D(A_L) = \{z \in D(L); (A+BL)z \in H\}, \end{cases} \tag{19}$$

is the infinitesimal generator of a strongly continuous semigroup $(\mathbb{S}(t))_{t\geq 0}$ on H and the origin of the following system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z = (A+BL)z, \\ z(0) = z_0, \end{cases} \tag{20}$$

is globally asymptotically stable.

Sliding surface

Let $\varphi \in D(A_L^*)$ is an eigenfunction of A_L^* such that $B^*\varphi \neq 0$ and λ the eigenvalue associated with φ. We define the sliding surface as follow Σ

$$\Sigma := \{z \in H \mid \langle \varphi, z \rangle_H = 0\}.$$

Its related sliding variable $\sigma : \mathbb{R}_+ \to \mathbb{K}$ is defined by

$$\sigma(t) := \langle \varphi, z(t) \rangle_H \tag{21}$$

for any solution z of (1).

We consider the control u to be

$$u = Lz + u_{SMC} \tag{22}$$

Formally, the derivative of σ along the trajectory of (1) and (22) yields,for all $t \geq 0$

$$\begin{aligned}
\dot{\sigma}(t) &= \langle \varphi, \frac{\mathrm{d}}{\mathrm{d}t} z(t) \rangle_H \\
&= \langle \varphi, A_L z(t) \rangle_H + B^* \varphi \big(u_{SMC}(t) + d(t) \big) \\
&= \langle A_L^* \varphi, z(t) \rangle_H + B^* \varphi \big(u_{SMC}(t) + d(t) \big) \\
&= \lambda \langle \varphi, z(t) \rangle_H + B^* \varphi \big(u_{SMC}(t) + d(t) \big), \quad \text{because } A_L^* \varphi = \lambda \varphi \\
\dot{\sigma}(t) &= \lambda \sigma(t) + B^* \varphi \big(u_{SMC}(t) + d(t) \big).
\end{aligned} \tag{23}$$

Sliding mode Control

We choose $u_{SMC}(t) = -\frac{1}{B^*\varphi}\lambda\sigma(t) + c_0(t)$. Then,

$$\dot{\sigma}(t) = B^*\varphi\big(c_0(t) + d(t)\big). \tag{24}$$

Thus, the following holds, for all $t \geq 0$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\sigma(t)|^2 &= \mathfrak{Re}\big(\bar{\sigma}(t)\dot{\sigma}(t)\big) \\ &= \mathfrak{Re}\Big(\bar{\sigma}(t)B^*\varphi\big(c_0(t) + d(t)\big)\Big). \end{aligned} \tag{25}$$

Assumption 2

The unknown disturbance d is supposed to be uniformly bounded measurable, i.e $|d(t)| \leq K_d$ for some $K_d > 0$ and for all $t \geq 0$.

Sliding mode Control

Therefore, we choose $c_0(t) = -\frac{K}{B^*\varphi}\text{sign}(\sigma(t)$, with $K > K_d|B^*\varphi|$ and sign is defined by

$$\text{sign}(s) = \begin{cases} \frac{s}{|s|} & \text{if } s \neq 0, \\ [-1, 1] & \text{if } s = 0. \end{cases}$$

Then, we have, for all $t \geq 0$

$$\frac{1}{2}\frac{d}{dt}|\sigma(t)|^2 \leq -(K - |B^*\varphi|K_d)|\sigma(t)|. \tag{26}$$

As a consequence, there exists a finite time $t_r > 0$, such that $\sigma(t) = 0$ for any $t \geq t_r$.

Conclusion

With the control input $u(t) = Lz(t) - \frac{1}{B^*\varphi}\Big(\lambda\sigma(t) + K\text{sign}(\sigma(t))\Big)$, the system (1) reaches the sliding surface Σ in finite time t_r and remains on it.

Sliding mode Control

Closed-loop system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z = A_L z + B\left(d - \frac{1}{B^* \varphi}\left(\lambda\sigma + K\mathrm{sign}(\sigma)\right)\right), \\ z(0) = z_0. \end{cases} \tag{27}$$

$$\forall t \geq t_r, \quad \sigma(t) = 0 \Longrightarrow \dot{\sigma}(t) = 0, \quad \forall t \geq t_r.$$

Thus, from (23), we have

$$c_0(t) + d(t) = 0, \quad \forall t \geq t_r$$

i.e

$$d(t) - \frac{K}{B^* \varphi}\mathrm{sign}(\sigma(t)) = 0, \quad \forall t \geq t_r.$$

Then, the system (27) become for any $t \geq t_r$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z = A_L z, \\ z(0) = z_0, \end{cases} \tag{28}$$

which is globally asymptotically stable around $(0, 0)$ from the item (iii) of Assumption 1.

Sliding mode Control

Theorem (Existence of closed-loop systems solutions)

Assume that Assumption 1 and Assumption 2 are satisfied. For any initial condition $z_0 \in H$, the system (27) admits a mild solution.

Theorem (Global asymptotic stability)

Assume that Assumption 1 and Assumption 2 are satisfied. For any initial condition $z_0 \in H$, $0 \in H$ is globally asymptotically stable for (27).

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Heat equation

$$\begin{cases} z_t(t,x) = z_{xx}(t,x), & (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z_x(t,0) = c_0 z(t,0), & t \in \mathbb{R}_+, \\ z_x(t,1) = u(t) + d(t), & t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \tag{29}$$

This equation can be written in an abstract way as in (1) if one sets $H = L^2(0,L)$,

$$\begin{aligned} A : D(A) \subset L^2(0,L) \to L^2(0,L), \\ z \mapsto z'', \end{aligned} \tag{30}$$

where

$$D(A) := \{z \in \mathcal{H}^2(0,1) \mid z'(0) = c_0 z(0); z'(1) = 0\}. \tag{31}$$

Heat equation

The control operator B is the delta function in $\mathcal{L}(\mathbb{R}, D(A)')$ defined as follow

$$\langle \varphi, Bu \rangle_{D(A),D(A)'} = \varphi(1)u \tag{32}$$

for all $u \in \mathbb{R}$ and $\varphi \in D(A)$, where $\langle \cdot, \cdot \rangle_{D(A),D(A)'}$ is the dual product. The adjoint operator of A is

$$\begin{aligned} A^* : D(A^*) \subset H &\to H, \\ z &\mapsto z'', \end{aligned} \tag{33}$$

with $D(A^*) := \{z \in \mathcal{H}^2(0,1) \mid z'(0) = c_0 z(0); z'(1) = 0\}$. It can be checked that the operator A is self-adjoint in H. The adjoint of operator of B is

$$\begin{aligned} B^* : D(A^*) &\to \mathbb{R} \\ \varphi &\mapsto \varphi(1). \end{aligned} \tag{34}$$

- (J-J. Liu and J-M Wang, 2015) The operators A and B satisfy the items (i) and (ii) Assumption 1.
- (J-J. Liu and J-M Wang, 2015) The origin of

$$\begin{cases} z_t(t,x) = z_{xx}(t,x), \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z_x(t,0) = c_0 z(t,0), \quad t \in \mathbb{R}_+, \\ z_x(t,1) = 0, \quad t \in \mathbb{R}_+, \\ z(0,x) = z_0(x), \end{cases} \tag{35}$$

is globally exponentially stable in H. Thus, Item (iii) of Assumption 1 holds for the operator L equal to the zero operator.

The eigenpairs $(\lambda, \varphi_\lambda)$ of A satisfies

$$\begin{cases} \lambda < 0, \\ \varphi_\lambda(x) = \cos(\sqrt{-\lambda}x) + \dfrac{c_0}{\sqrt{-\lambda}}\sin(\sqrt{-\lambda}x), \\ \sqrt{-\lambda}\tan(\sqrt{-\lambda}) = c_0. \end{cases} \tag{36}$$

The sliding variable and the feedback law under consideration are as follows

$$\sigma(t) = \int_0^L z(t,x)\varphi_\lambda(x)dx \quad \text{and} \quad u(t) = -\frac{1}{\varphi_\lambda(1)}\big(\lambda\sigma(t) + K\text{sign}(\sigma(t))\big). \tag{37}$$

Heat equation

- $c_0 = 0.5,$
- $\lambda = -2c_0 - \pi^2$
- $K = 2.5$
- $z_0(x) = 10x^3$
- $d(t) = 2\sin(t)$