Inverse Eigenvalue Problems in Control and Observation

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Introduction

The *pole placement problem* is the most standard inverse problem in control. We consider a linear, finite dimensional system

$$\dot{x}(t) = Ax(t) + Bu(t) \ ,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, for some integers n and m, and $x(t)$ is a state vector, $u(t)$ a control vector, both function of the time t. A feedback control law of the form

$$u = Fx(t) \ ,$$

where $F \in \mathbb{R}^{m\times n}$, is applied, leading to the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \ .$$

We are interested into the poles of the closed-loop system –that are nothing but the eigenvalues of the matrix $A + BF$. If F is given, one can calculate them.

Calculating F to obtain a given set of eigenvalues was the first inverse problem.

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Historical Notes

This formulation is a well-known history :

– Bellman (1957) and other ones have shawn that the solution of optimal control problems was in the form of a static state feedback

– Kalman (1960) has extended the concept to estimation problems, and linked the controllability and the ability to freely assign the poles of a system

– Wonham (1969) and other ones have provided various algorithms to calculate F, using state-space methods.

– Rosenbrock (1970) used polynomial methods to address the question.

Rosenbrock also shown that the problem is not so simple:

Controllability permits to freely assign the poles location, but the multiplicity of the poles cannot be freely assigned, in general.

Fundamental Rosenbrock's Theorem on Pole Placement

Be given the pair (A, B) as above, that is assumed to be controllable, and a set of polynomials $\alpha_i(s)$, there exists a feedback F so that the invariant factors of $sI_n - A - BF$ are the $\alpha_i(s)$ if and only if

$$\sum_{i=1}^{k} \deg \alpha_i(s) \geq \sum_{i=1}^{k} \delta_i \ , \ \text{for } k = 1 \text{ to } n \ ,$$

with equality for $k = n$,
– where of course the polynomials $\psi_i(s)$ are monic polynomials satisfying $\alpha_{i+1}(s)|\alpha_i(s)$, for $i = 1$ to $n - 1$,
– and where the integers δ_i are the controllability indices of (A, B).

This precisely expresses that the pole location can be freely assigned, but their multiplicities are constrained.

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Outline

We shall discuss this old result, to point out :

- various interpretations in terms of systems transformations,
- extensions to other assignment problems,
- recent publications by Kučera, that motivate to come back to the question,
- some open problems.

To this aim, we first recall some definitions and notations.

Kronecker normal form (Gantmacher, 1959)

The Kronecker normal form $sE_K - A_K$ of a matrix pencil $sE - A$, is obtained using invertible and constant changes of bases P, Q

$$(E, A) \longrightarrow (PEQ, PAQ)$$

$sE_K - A_K$ consists of diagonal blocks, that are of four types, and are associated to

- the finite invariant factors $\phi_i(s)$,
- the infinite zero orders n_i,
- the column minimal indices c_i,
- the row minimal indices r_i.

It was initially introduced by Kronecker (1866, 1868).

Kronecker normal form (cont.)

The four types of blocks that appear in the Kronecker normal form $sE_K - A_K$, of a pencil $sE - A$, have the following structure :

$$(fif) \quad \left.\begin{bmatrix} s-\lambda_j & -1 & & \\ & \ddots & \ddots & \\ & & s-\lambda_j & -1 \\ & & & s-\lambda_j \end{bmatrix}\right\} n_{ij} \qquad (izo) \quad \left.\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}\right\} n_i$$

$$(cmi) \quad \left.\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}\right\} c_i \qquad (rmi) \quad \left.\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}\right\} r_i + 1$$

The finite zeroes of the pencil are the number λ_i, that are the zeroes of the invariant factors $\psi_i(s) = \prod_j (s-\lambda_j)^{n_{ij}}$. The polynomials $(s-\lambda_j)^{n_{ij}}$ are called the elementary divisors of the pencil.

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Pole, Zeroes, Controllability Indices

The invariant factors of A, mentioned into Rosenbrock's Theorem, are that of the pencil

$$sI_n - A \ .$$

They describe the pole structure of the system (locations and multiplicities).

The zeroes of a linear system with output equation $y = Cx + Du$ are those of the pencil (called system matrix)

$$\begin{bmatrix} sI_n - A & -B \\ C & D \end{bmatrix} .$$

The invariant factors of this pencil describe the zero structure of the system (their locations and multiplicities).

The controllability indices of a pair (A, B) are nothing but the column minimal indices of the input pencil

$$\begin{bmatrix} sI_n - A & -B \end{bmatrix} .$$

More Transformation Groups and Invariants

The set of invertible and constant matrices P, Q that are used to define the Kroneker normal form is a transformation group that acts on the set of linear system

$$(E, A) \longrightarrow (PEQ, PAQ) \ .$$

In the same way, one uses unimodular matrices $U(s)$ and $V(s)$ to define the Smith form of a polynomial matrix $P(s)$, or the Smith-McMillan form of a rational matrix

$$P(s) \longrightarrow U(s)P(s)V(s) \ ,$$

and uses biproper matrices $S(s)$, $T(s)$ to get the Smith form at infinity of a rational matrix $R(s)$

$$R(s) \longrightarrow S(s)R(s)T(s) \ .$$

Finally the left Wiener-Hopf form is obtained using both biproper and unimodular operations of a matrix $R(s)$

$$R(s) \longrightarrow S(s)R(s)U(s) \ .$$

Polynomial and Rational canonical forms

The Smith form, Smith McMillan form, Smith form at infinity, left Wiener-Hopf form have the same structure. They are diagonal modulo a completion by null columns or rows

$$\begin{bmatrix} \mathrm{diag}\{\star\} & 0 \\ 0 & 0 \end{bmatrix} .$$

The diagonal elements are respectively polynomials $\alpha_i(s)$, rational fractions $\frac{\epsilon_i(s)}{\psi_i(s)}$, integers n_i, and terms s^{c_i}.

The froms are uniquely defined if the diagonal elements are taken in non-increasing order, say

$$\alpha_{i+1}(s)|\alpha_i(s) \ ,$$
$$\epsilon_{i+1}(s)|\epsilon_i(s) \ , \text{ and } \psi_i(s)|\psi_i(s) \ ,$$
$$n_{i+1} \leq n_i \ ,$$
$$c_{i+1} \leq c_i \ .$$

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Pole, Zeroes, Controllability Indices

As it is well-known (from Rosenbrock (1970), see also Kailath (1980)), there are relationships between the invariants.

The transfer $T(s) = C(sI_n - A)^{-1}B$ can be factored in the form $T(s) = N(s)D^{-1}(s)$.

If the system is controllable and observable, then the non-unit invariant factors of $N(s)$ are those of the system matrix, and the ones of $D(s)$ are those of $sI_n - A$.

They are respectively equal to the polynomials $\epsilon_i(s)$ and $\psi_i(s)$ that are at the numerator and denominator of the Smith-McMillan form of thye transfer $T(s)$.

The controllability indices are the degrees c_i of the columns of the matrix $D(s)$, say its left Wiener-Hopf indices.

We can now come back to the various interpretations of Rosenbrock's Theorem

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Initial Pole Placement Theorem – Rosenbrock (1970)

To begin with, let us remark that the above formulation is actually due to Kučera (1979).

The initial version from Rosenbrock was presenting necessary conditions, and sufficient ones, in a somehow more involved way.

4.2 Assignment of closed-loop poles

The second question posed at the beginning of Section 4 is answered by the following theorem.

Theorem 4.2 With A, B, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ as in Theorem 4.1, the $m \times n$ matrix C may be chosen, relatively (right) prime to $sI_n - A$, so that the McMillan form of $H(s) = G(s)[I_m + G(s)]^{-1}$ is diag $(\varepsilon_i(s)/\psi_i'(s))$, where

(i) the ε_i are the numerator polynomials in the McMillan form diag $(\varepsilon_i(s)/\psi_i(s))$ of $G(s)$;

(ii) the ψ' are any prescribed non-zero monic polynomials subject to the conditions (of which the first two are necessary);

(a) for $i = 2, 3, \ldots, m$, ψ_i' divides ψ_{i-1}',

(b) the degrees $\delta(\psi_i')$ of the ψ_i' satisfy

$$\sum_{i=1}^{k} \delta(\psi'_{m+1-i}) \leqslant \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, m \qquad (4.36)$$

with equality holding for $k = m$,

(c) ψ_1' and ψ_1 are relatively prime.

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Another Rosenbrock's Theorem, on Zero Placement

The previous Theorem is preceded by the following (Rosenbrock, 1970, Page 186):

4.1 Assignment of open-loop zeros

We shall now prove the following theorem.

Theorem 4.1 Let the $n \times n$ matrix A and the $n \times m$ matrix B be given, with $sI_n - A$ and B relatively (left) prime. Let the minimal indices of $(sI_n - A\ B)$ in order of magnitude be $\lambda_1, \lambda_2, \ldots, \lambda_m$, with $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-q} = 0$, $q \leqslant m$. Then the $m \times n$ matrix C may be chosen, relatively (right) prime to $sI_n - A$, so that the McMillan form of $G(s) = C(sI_n - A)^{-1}B$ is diag $(\varepsilon_i(s)/\psi_i(s))$, where

(i) the ψ_i are determined by the Smith form diag $(I_{n-m}, \psi_m, \psi_{m-1}, \ldots, \psi_1)$ of $sI_n - A$, in which $\psi_m = \psi_{m-1} = \cdots = \psi_{t+1} = 1, t \leqslant q$;

(ii) the ε_i may be any prescribed monic polynomials subject to the conditions (of which the first three are necessary)

(a) $\varepsilon_{r+1} = \varepsilon_{r+2} = \cdots = \varepsilon_m = 0$, $t \leqslant r \leqslant q$

(b) ε_i divides ε_{i+1}, $i = 1, 2, \ldots, r-1$

(c) the degrees $\delta(\varepsilon_i)$ of the non-zero ε_i satisfy

$$\sum_{i=1}^{k} \delta(\varepsilon_i) \leqslant \sum_{i=1}^{k} (\lambda_{m-r+i} - 1), \quad k = 1, 2, \ldots, r \tag{4.11}$$

(d) $r = q$, and ε_q, ψ_1 are relatively prime.

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Two Different Problems – A Unique Formulation

Rosenbrock had actually mixed pole and zero placement because of their analogy.

Assigning the poles of $A + BF$ comes down to complete the initial input pencil by the rows corresponding to the control law, so that the resulting pencil

$$\begin{bmatrix} sI_n - A & -B \\ -F & I_n \end{bmatrix},$$

has a given set of invariant polynomials.

Assigning the zeros by choice of the matrices of the output equation $y = Cx + Du$ comes down to a similar problem: that of assigning the invariant factors obtained by row completion of the input pencil that is given

$$\begin{bmatrix} sI_n - A & -B \\ C & D \end{bmatrix}.$$

Both problems are inverse row matrix pencil completion problems.

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Taking rank condition into account

Actually, since D is not invertible in general, there is a slight difference between both problems and between the solutions.

Be given a controllable system and a set of monic polynomials ranged in non-increasing order, we have precisely the following.

There exists F such that the closed-loop system has the pole structure described by the polynomials $\alpha_i(s)$ if and only if

$$\sum_{i=1}^{k} \deg \alpha_i(s) \geq \sum_{i=1}^{k} \delta_i \ , \ \text{for } k = 1 \text{ to } n \ ,$$

with equality for $k = n$.

There exists matrices c and D such that the zero structure of the system matrix is given by the polynomials $\alpha_i(s)$ if and only if

$$\sum_{i=k}^{n} \deg \alpha_i(s) \leq \sum_{i=k}^{n} \delta_i \ , \ \text{for } k = 1 \text{ to } n$$

without required equality for $k = n$.

Both inequalities are equivalent if equality holds for $k = n$. The second one is NSC for the row completion problem by constant matrices.

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Dual problems – Matrix completion

One can dualy associate Rosenbrock's Theorem to a column completion problem.

The matrix A being given, with invariant factors $\alpha_i(s)$, for $i = 1$ to n, there exists a matrix B such that the controllability indices of the pair (A, B), say the column minimal indices of the input pencil $[sI - n - A, -B]$, are equal to δ_i, for $i = 1$ to m, if and only if

$$\sum_{i=1}^{k} \deg \alpha_i(s) \geq \sum_{i=1}^{k} \delta_i \ , \ \text{for } k = 1 \text{ to } n \ .$$

In addition, the resulting system is controllable if equality holds for $k = n$.

In some sense, this characterization in terms of column completion is more naturally associated to the pole placement problem. The row completion problem was actually obtained thanks to the additional equality.

Polynomial Transformations

Suppressing rows or columns to a matrix pencil can be seen as non-regular transformations. We can also reinterpret Rosenbrock's theorem in such terms.

Be given a polynomial $m \times m$ matrix $P(s)$ with minimal column degrees c_i, $i = 1$ to m, there exists a biproper matrix $B(s)$ such that the product $B(s)P(s)$ is polynomial with invariant factors $\alpha_i(s)$, $i = 1$ to m, if and only if Rosenbrock's conditions hold true.

Indeed, if we write $(sI_n - A)^{-1}B = N(s)D^{-1}(s)$, then the system looped by the feedback $u = Fx + Gv$, with G invertible, leads to the closed-loop transfer $N(s)(D(s) - FN(s))^{-1}G$, showing that the closed-loop denominator is equal to $B(s)D(s)$, with $B(s) = G^{-1}(I_m - FN(s)D^{-1}(s))$.

Reversely, a precompensator is realizable by static state feedback if and only if it is biproper, say $B^{-1}(s)$, and such that $B(s)D(s)$ is polynomial.

In the same way, one can change the minimal column degrees of a polynomial matrix by left multiplication by a unimodular matrix, to get them equal to c_i, if and only if its invariant factors satisfy the Rosenbrock's conditions.

Transformation monoids define order relations

A transformation monoid $\mathcal{M}$ acting on a set $\mathcal{E}$ is a set of transformations:

$$m \in \mathcal{M} : \mathcal{E} \longrightarrow \mathcal{E}$$

endowed with a composition law that is associative and admits a neutral element.

The relation $\mathcal{R} \subset \mathcal{E} \times \mathcal{E}$, defined by: $\forall e, f \in \mathcal{E}, e\mathcal{R}f \iff \exists m \in \mathcal{M} \,|\, me = f$, is a preorder, since

– it is transitive because the composition is associative,

– and it is reflexive because a neutral element exists.

The relation $\sim$, $\forall e, f \in \mathcal{E}, e \sim f \iff \exists m, p \in \mathcal{M} \,|\, me = f, pf = e$, is, in turn an equivalence.

The relation $\bar{\mathcal{R}}$, induced by $\mathcal{R}$ into the quotient $\mathcal{E}/\sim$ is always an order relation.

In the case of Rosenbrock's Theorem, the order relation between the equivalent classes is explicitely characterized by inequalities between the invariants characterizing the classes.

Other Such Results – Sà & Thompson Theorem

It concerns the row or column completion of polynomial matrices.

The result is easier to express taking the invariant factors with degrees in increasing order:

$$\alpha_i(s)\,|\alpha_{i+1}(s),\ \text{for}\ i \geq 1\ ,$$

where by convention, $\alpha_i(s) = 0$ if i is greater than the rank of the considered matrix.

(Sà & Thompson, 1979) Be given a matrix $A(s) \in \mathbb{R}^{p\times q}[s]$, with invariant factors $\alpha_i(s)$, there exists a polynomial matrix $P(s) \in \mathbb{R}^{p\times r}[s]$ such that the composite matrix has for invariant factors a given set of polynomials $\beta_i(s)$ (that are monic and ranged in increasing order), if and only if $\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+r}(s)$, for $i = 1$ to p (taking $\alpha_i(s)$=0 if $i > \text{rank}\, A(s)$).

For instance if $p = 2$ and $q = 1$, the matrix $A(s)$ has a unique invariant factor $\alpha(s)$. The matrix completed by a second column can have rank 1 or 2.

In the first case, the unique invariant factor of the completed matrix is $\beta(s) \mid \alpha(s)$.

In the second case, the invariant factors $\beta_1(s)$ and $\beta_2(s)$ satisfy $\beta_1(s) \mid \alpha(s) \mid \beta_2(s)$

Zaballa Theorem

Mixing Rosenbrock's Theorem, and Sà & Thompson's result, the pole placement for non-controllable systems was obtained by Zaballa (1987)

Be given an (A, B) pair with controllability indices c_i and non-controllable pole structure given by the invariant factors $\alpha_i(s)$, $i = 1$ to q, ranged in non-decreasing order, and given a set $\beta_i(s)$, $i = 1$ to n, of monic polynomials, also ranged in non-decreasing order, then it exists a feedback F such that the invariant factors of the closed-loop matrix $A + BF$ are the β_i if and only if the following conditions hold true:

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+m}(s) \ ,$$

$$\sum_{i=1}^{q} \deg \alpha_i(s) + \sum_{i=1}^{k} c_{m-i+1} \geq \sum_{i=1}^{q+k} \deg \beta_i^k(s) \ , \ \text{for } k = 1 \text{ to } m \ ,$$

with equality for $k = m$, and where, by definition:

$$\beta_i^k(s) = \text{lcm}(\alpha_{i-k}(s), \beta_i(s)) \ , \ \text{for } k = 1 \text{ to } m \ , \ \text{and } i = 1 \text{ to } q + k \ .$$

Feedback Simulation I

(Heymann, 1976) Be given an (A, B) pair with controllability indices c_i, $i = 1$ to m, there exists matrices F and G such that the controllability indices of $(A + BF, BG)$ are c'_i, $i = 1$ to m', if and only if

$$\sum_{j \mid c'_j \le i} c'_j \le \sum_{j \mid c_j \le i} c_j \ , \ \text{for } i = 1 \text{ to } m \ .$$

This relates the minimal column indices of a pencil and of a subpencil.

As a typical example, if the controllability indices are 3 and 1, then it is impossible to have an invariant factor of degree 2 for the closed-loop system. More generally, the dimensions of the controllability subspace are in a subset of the interval $[0, n]$. Some values are impossible.

Assignment of the Transmission Pole Structure

As above, the feedback can alter the controllability and the transmission pole structure.

(With Zagalak and Kučera, 1999) Be given an (A, B) pair, and a list of monic polynomials $\psi_i(s)$, ranged in non-increasing order, there exists an $m \times n$ matrix F and an $m \times p$ matrix G such that the pole structure of the closed-loop transfer, say the set of denominators of the diagonal elements of the Smith-McMillan form of $(sIn - A - BF)^{-1}BG$, is given by the polynomials $\psi_i(s)$, if and only if it exists a list of m non negative integers $c'i$, $j = 1$ to m, ranged in non-increasing order, such that

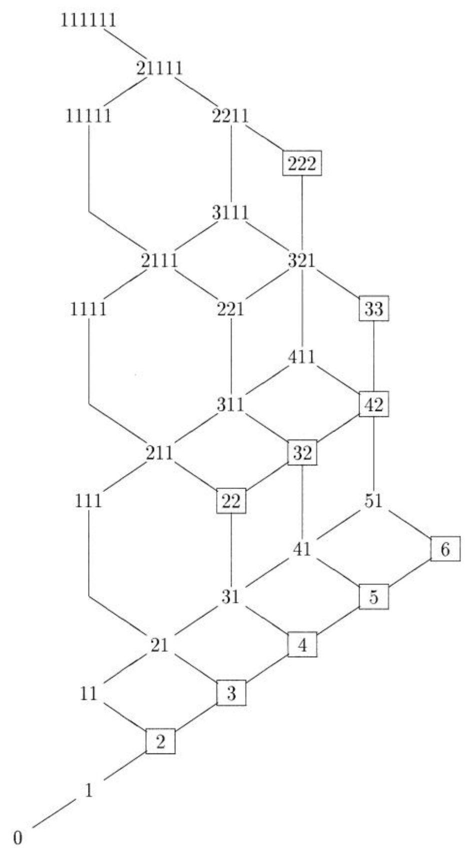
$$\sum_{i=k}^{n} \deg \psi_i(s) \leq \sum_{i=k}^{n} c'_i \ , \text{ for } k = 1 \text{ to } n \ ,$$

with equality for $k = 1$, and

$$\sum_{j|c'_j \leq i} c'_j \leq \sum_{j|c_j \leq i} c_j \ , \text{ for } i = 1 \text{ to } m \ .$$

Assignment of the Transmission Pole Structure (Cont.)

The conditions that are obtained are implicit, but they are checkable, since the first set of inequalities defines a lattice, that is finite, which solution set is a sublattice. The lattice obtained for $n = 6$ and the sublattice corresponding to $c = (2, 2, 2, 0, 0, 0)^T$ is the following:



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Some More Results

There are also a paper from Loiseau (1986), concerning the assignment by feedback of the infinite zero orders of a given (C, A, B) triple, knowing the infinite zero orders and column minimal indices of this triple.

Baragaña obtained a characterization of the possible controllability indices and invariant factors of a pair $(A + BF, BG)$ obtained via a possibly non-regular static state feedback.

Finally, Mondié (1997) characterized the relationships between two right invertible pencils, when the first one is a sub-pencil of the second one.

This was the following statement.

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Theorem 1. *Given $\lambda E - H$ and $\lambda E' - H'$ two right-invertible matrix pencils, let $c_1 \geqslant c_2 \geqslant \cdots c_{p+q-m}$ and $c'_1 \geqslant c'_2 \geqslant \cdots \geqslant c'_{p-m}$, respectively, denote their column minimal indices, $\alpha_1(\lambda) \mid \alpha_2(\lambda) \mid \cdots \mid \alpha_m(\lambda)$ and $\alpha'_1(\lambda) \mid \alpha'_2(\lambda) \mid \cdots \mid \alpha'_m(\lambda)$ their invariant factors, $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{m-r}$ and $n'_1 \geqslant n'_2 \geqslant \cdots \geqslant n'_{m-r}$ their infinite zero orders. Then there exists a matrix pencil $F_{12}(\lambda)$ so that $\lambda E - H \sim (\lambda E' - H', F_{12}(\lambda))$ if and only if the following conditions hold:*

$$\alpha_i(\lambda) \,|\, \alpha'_i(\lambda) \,|\, \alpha_{i+q}(\lambda), \quad i = 1, \ldots, m,$$

$$n'_{i+q} \leqslant n_i \leqslant n_i{}', \quad i = 1, \ldots, m-r,$$

$$c'_{i+q} \leqslant c'_i, \quad i = 1, \ldots, p-m,$$

$$\sum_{i=1}^{l_j} c_i + \sum_{i=1}^{m} \deg \operatorname{lcm}\left(\alpha_i(\lambda), \alpha'_{i-j}(\lambda)\right) + \sum_{i=1}^{m-r} \max\left(n_i, n'_{i+j}\right) \leqslant \sum_{i=1}^{l_j - j} c'_i + \sum_{i=1}^{m} \deg \alpha'_i(\lambda) + \sum_{i=1}^{m-r} n'_i, \quad j = 1, \ldots, q,$$

$$\sum_{i=1}^{p+q-m} c_i + \sum_{i=1}^{m} \deg \alpha_i(\lambda) + \sum_{i=1}^{m-r} n_i = \sum_{i=1}^{p-m} c'_i + \sum_{i=1}^{m} \deg \alpha'_i(\lambda) + \sum_{i=1}^{m-r} n'_i,$$

where, by definition

$$l_j = \min\left\{ i \,|\, c'_{i-j+1} < c_i \right\}, \quad j = 1, \ldots, q$$

and, by convention, $\alpha'_i(\lambda) = 1$ for $i \leqslant 0$, and $n_i = 0$ for $i \geqslant m - r$.

It should be emphasized that the conditions are checkable. The problem is decidable.

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Recent Contributions

There are still regular publications on the thema, see in particular:

— Dodig (2022) Matrix pencils completions under double rank restrictions
— Dodig and Stosic (2021) Completion of matrix pencils with a single rank restriction
— Baragaña and Roca (2021), The change of the Weierstrass structure under one row perturbation
— Dodig and Stosic (2020) Rank One Perturbations of Matrix Pencils
— Baragaña, Dodig, Roca, and Stosic (2020) Bounded rank perturbations of regular pencils over arbitrary fields
— Baragaña and Roca (2020)Rank-one perturbations of matrix pencils
— Dodig and Stosic (2019) The General Matrix Pencil Completion Problem: A Minimal Case
— Gernandt and Trunk (2017) Eigenvalue placement for regular matrix pencils with rank one perturbations
etc.

Recent Contributions

Another series of recent publications comes from Kučera emphasize the link with classical control problems, which renews the interest for the topic:

— Kučera (2022) Assignment of infinite zero orders using state feedback
— Kučera (2020) Decoupling With Stability by Static-State Feedback
— Kučera (2018) Block decoupling of linear systems by static-state feedback
— Kučera (2018) Assignment of Invariant and Transmission Zeros in Linear Systems
— Kučera (2017) Model matching by dynamic state feedback
— Kučera (2017) Diagonal decoupling of linear systems by static-state feedback
— Kučera (2016) Stable model matching by non-regular static state feedback

Let us recall that the model matching problem consists in finding a feedback (F, G) such that the closed loop transfer is equal to a prescribed model: $C(sI_n - A - BF)^{-1}BG = T_m(s)$.

The decoupling is to find (F, G) with a diagonal model $T_m(s)$.

Kučera proposes semi-algorithmic procedures, partly derived from the completion methods.

Feedback Simulation II

There is a control problem that is completely reduced to a completion problem.

(with Zagalak, 1994) Be given an (A, B) pair with controllability indices c_i, $i = 1$ to m, there exists matrices C, F and G such that $C(sI_n - A - BF)^{-1}BG = T_m(s)$, if and only if

$$\sum_{j|c'_j \leq i} c'_j \leq \sum_{j|c_j \leq i} c_j \ , \ \text{for } i = 1 \text{ to } m \ .$$

where c'_i, $i = 1$ to p are the minimal column degrees of a denominator of $T_m(s)$

This is a variant of Heymann's result, presented page 20. The result is actually constructive, leading to an effective method to compute a solution C, F, and G.

We present hereafter Kučera's method to address the model matching by staic state feedback, which is not fully constructive.

Model Matching by Static State Feedback

(Kučera 2015) Be given a $p \times m$ system (C, AB, D), and a model of transfer $T_m(s)$ of size $p \times r$, there exists a feedback (F, G) such that

$$D + C(sI_n - A - BF)^{-1}BG = T_m(s) \ ,$$

if and only if there exists an $m \times r$ polynomial matrix $U(s)$, an $r \times r$ polynomial and non-singular matrix $V(s)$, an $m \times r$ constant matrix Z, and an $m \times m$ polynomial matrix, column reduced with column degrees equal to d_i, $i = 1$ to m, such that

$$\begin{bmatrix} ZD_m(s) \\ N_m(s) \end{bmatrix} V(s) = \begin{bmatrix} H(s) \\ N(s) \end{bmatrix} U(s) \ ,$$

where $D + C(sI_n - A)^{-1}B = N(s)D^{-1}(s)$, $N(s)$ and $D(s)$ are right coprime and $D(s)$ is column reduced with column degrees equal to d_i, $i = 1$ to m, and $T_m(s) = N_m(s)D_m^{-1}(s)$, $N_m(s)$ and $D_m(s)$ are right coprime.

A key point is to complete $N(s)$ by a matrix that is column reduced.
There are other such formulations in the litterature, where the completion problem is subject to additional constraints.

Diagonal Decoupling by Static State Feedback

Kučra (2017) stated the following implicit solution, in the spirit of the transmission pol assignment, to the diagonal decoupling problem by static state feedback.

Theorem 5: Given a right-invertible system (A, B, C, D) with $\rho := (\rho_1, \rho_2, \ldots, \rho_p)$ the list of infinite zero orders and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_{m-p})$ the list of column minimal indices. Let $\Phi(s)$ be the interactor of the system and $\varphi := (\varphi_1, \varphi_2, \ldots, \varphi_p)$ the list of essential orders.

The system is decouplable by static-state feedback if and only if there exist three lists of nonnegative integers, $\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$, $\eta := (\eta_1, \eta_2, \ldots, \eta_{m-p})$, and $\eta^* := (\eta_1^*, \eta_2^*, \ldots, \eta_p^*)$, where η^* is a selection of the elements of η satisfying $J_{\eta^*} = J_\varepsilon$, such that the following conditions are all satisfied:

$$\varepsilon + \rho \geq \varphi \tag{17}$$

$$\text{card } J_\omega \leq m - p \tag{18}$$

$$\varepsilon + \rho \geq \eta^* \tag{19}$$

$$\eta \leq \sigma \tag{20}$$

$$\eta \succ \omega \tag{21}$$

$$\text{sum } \varepsilon = \text{sum } \eta = \text{sum } \omega \tag{22}$$

where $\omega := (\omega_1, \omega_2, \ldots, \omega_p)$ is the list of the orders of the invariant factors $s^{-\omega_1}, s^{-\omega_2}, \ldots, s^{-\omega_p}$ of $\Phi(s)\Lambda_{\varepsilon+\rho}^{-1}(s)$.

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Some open problems

As a conclusion, we highlight some open issues.

— Assess the decidability of Kucera formulations of model matching and decoupling, and of other such control problems, is a renewed and promising direction.

— A second problem is the completion of general matrix pencils. Mondié (1997) has solved the case of right-invertible pencils. By duality, we can also solve the case of left invertible pencils, but the general case is open, since 1998 (LAA challenge).

— All the results can also be applied to observer synthesis as well, to find NSC for the pole assignment of $A + KC$ and other similar problems. An important open problem is to give NSC for the pole placement by dynamic feedback, say $u = Fw$, with $\dot{w} = Aw + Bu + K(Cw - Cx)$. The closed-loop extended matrix is

$$\begin{bmatrix} sI_n - A - BF & -BF \\ 0 & sI_n - A - KC \end{bmatrix}.$$

The evaluation of the multiplicities of its zeros is perturbed by the right upper term. This is the general problem of pole assignment (posed by Rosenbrock & Hayton, 1978).

THANK YOU FOR YOUR ATTENTION !