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Logaritmic Convexity of Semigroups and Inverse Problems of Ornstein-Uhlenbeck equations

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Inverse initial data problems

We study the **inverse problem of determining initial data** of the well-posed linear system

$$\begin{cases} u'(t) = Au(t), & t \in (0, \theta], \\ u(0) = u_0 \in H, \end{cases}$$

from the observations

$$v(t) = \mathbf{C}u(t), \qquad t \in (0, \theta].$$

$\mathbf{C} \in \mathcal{L}(D(A), Y)$ is an observation operator for $(\mathrm{e}^{tA})_{t \geq 0}$.

Logarithmic stability

We aim to show a **logarithmic stability** estimate for a class of initial data:

$$\|u_0\|_H \leq \frac{C}{\left(-\log \|\mathbf{C}u\|_{L^2(I,Y)}\right)^{\alpha}},$$

for some $\alpha \in (0,1]$.

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General idea:

Observability inequality + Logarithmic convexity $\Longrightarrow$ Logarithmic stability

Observability

- The observation operator **C** is **admissible** if $\exists \kappa_\theta > 0$:

$$\forall u_0 \in D(A), \quad \int_0^\theta \left\| \mathbf{C} e^{tA} u_0 \right\|_Y^2 \mathrm{d}t \leq \kappa_\theta^2 \left\| u_0 \right\|_H^2 .$$

- The system is **final state observable** in time θ if $\exists \kappa_\theta > 0$:

$$\forall u_0 \in D(A), \quad \left\| e^{\theta A} u_0 \right\|_H^2 \leq \kappa_\theta^2 \int_0^\theta \left\| C e^{tA} u_0 \right\|_Y^2 \mathrm{d}t.$$

Logarithmic convexity

Consider the abstract parabolic system

$$\begin{cases} u'(t) = Au(t), & t \in (0, \theta], \\ u(0) = u_0 \in H. \end{cases}$$

- $\theta > 0$ is a final time for the system.
- $A : D(A) \subset H \to H$ is the generator C_0-semigroup $\left(\mathrm{e}^{tA}\right)_{t \geq 0}$.

Self-adjoint case

Lemma (Agmon-Nirenberg (1963))

*Assume that A is **self-adjoint**. The solution u satisfies*

$$\|u(t)\| \leq \|u_0\|^{1-\frac{t}{\theta}} \|u(\theta)\|^{\frac{t}{\theta}}$$

for all $0 \leq t \leq \theta$.

Key ideas: Differentiate $\log \|u(t)\|$ twice with respect to t and use symmetry and Cauchy-Schwartz inequality.

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Key ideas: Differentiate $\log \|u(t)\|$ twice with respect to t and use symmetry and Cauchy-Schwartz inequality.

Remark:

A function $f(t)$ that is $C^2[0,\infty)$ is log-convex if and only if the differential inequality

$$f(t)f''(t) - (f'(t))^2 \geq 0 \qquad (1)$$

holds for all $t \geq 0$.

Proof.
Since $D(A^2)$ is dense in H, it suffices to consider $u_0 \in D(A^2) \setminus \{0\}$.
We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\langle u'(t), u(t)\rangle = 2\langle Au(t), u(t)\rangle,$$

and since A is self-adjoint,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2 = 4\|Au(t)\|^2.$$

It follows that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2\right)\|u(t)\|^2 - \left(\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\right)^2 = 4(\|Au(t)\|^2\|u(t)\|^2 - \langle Au(t), u(t)\rangle^2).$$

By Cauchy-Schwarz inequality, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2\right)\|u(t)\|^2 - \left(\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\right)^2 \geq 0, \qquad 0 \leq t \leq \theta. \qquad (2)$$

Then

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t)\|^2\right)\|u(t)\|^2-\left(\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2\right)^2=\|u(t)\|^4\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log\left(\|u(t)\|^2\right)\geq 0. \tag{3}$$

Therefore, the function $t\mapsto\log\|u(t)\|$ is convex on $[0,\theta]$. We obtain

$$\|u(t)\|\leq\|u_0\|^{1-\frac{t}{\theta}}\|u(\theta)\|^{\frac{t}{\theta}}$$

for all $0\leq t\leq\theta$.

Remarks

- Logarithmic convexity

$$\|u(t)\| \leq K\|u_0\|^{1-\frac{t}{\theta}}\|u(\theta)\|^{\frac{t}{\theta}}$$

implies the **backward uniqueness** for the solution: if $u(\theta) = 0$, then $u_0 = 0$.

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- Logarithmic convexity holds for group of isometries.

- A well-posed problem need not satisfy logarithmic convexity:

$u_t + u_x = 0, \quad u(t, 0) = 0, \quad u(0, x) = u_0$, where $t \in (0, \theta), x \in (0, 1)$.

Stability estimate

- 2001: M. YAMAMOTO, J. ZOU, logarithmic stability for **initial data** in **heat** equation by **logarithmic convexity** and **observability inequality**.

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- 2009: J. LI, M. YAMAMOTO, J. ZOU, stability and **numerical reconstruction** of initial data for a **general parabolic equation**.
- 2011: G. GARCÍA, T. TAKAHASHI, Logarithmic stability for abstract **self-adjoint** dissipative operators.

Stability estimate in the self-adjoint case

For fixed $\varepsilon \in (0,1)$ and $M > 0$, consider

$$\mathcal{I}_{\varepsilon,M} := \left\{ u_0 \in D\left((-A)^{\varepsilon}\right) : \|u_0\|_{D((-A)^{\varepsilon})} \leq M \right\}.$$

Theorem (García-Takahashi (2011))

We assume that $u_0 \in \mathcal{I}_{\varepsilon,M}$ and the system is final state observable in time $\theta > 0$. For $p \in \left(1, \frac{1}{1-\varepsilon}\right)$ and $s \in \left(0, 1-\frac{1}{p}\right)$, $\exists K > 0$:

$$\|u_0\|_H \leq K \left(-\log \|\mathbf{C}u\|_{L^2(0,\theta;Y)}\right)^{-\frac{s}{p}},$$

provided that $\|\mathbf{C}u\|_{L^2(0,\theta;Y)} < 1$.

Analytic Non necessarily self-adjoint semigroups

Let A be the generator of a bounded **analytic semigroup** $(\mathrm{e}^{tA})_{t\geq 0}$ of angle $\psi \in \left(0, \dfrac{\pi}{2}\right]$. Set

$$\Sigma_\psi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \psi\},$$

and let $K \geq 1$ and $\kappa \geq 0$:

$$\|\mathrm{e}^{zA}\| \leq K \mathrm{e}^{\kappa \mathrm{Re}\, z} \quad \text{on } \overline{\Sigma}_\psi.$$

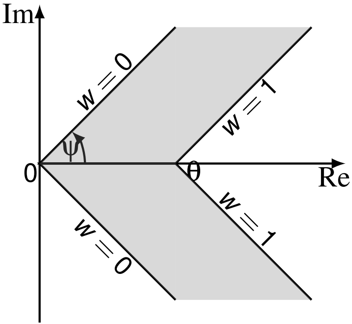
General logarithmic convexity

Theorem (Krein-Prozorovskaya (1960))

Let $u_0 \in H$. The solution satisfies

$$\|u(t)\| \le K e^{\kappa(t - \theta w(t))} \|u_0\|^{1-w(t)} \|u(\theta)\|^{w(t)}, \qquad 0 \le t \le \theta,$$

where w is the harmonic function on the strip $\mathcal{S}_\psi$



The harmonic function

The harmonic function w is given by

$$w(z) = \frac{\mathrm{Re}\left(h^{-1}(z)\right)}{\theta}, \quad h(z) = f \circ g(z), \quad g(z) = \theta \sin^2\left(\frac{\pi z}{2\theta}\right)$$

and

$$f(z) = \frac{\theta \sin\psi}{\pi} \int_0^{\frac{z}{\theta}} t^{\frac{\psi}{\pi}-1}(1-t)^{-\frac{\psi}{\pi}} \mathrm{d}t.$$

The harmonic function

In case $\psi = \frac{\pi}{2}$,

$$w(z) = \frac{t}{\theta}, \quad \forall z = t + is \in \mathcal{S}_{\frac{\pi}{2}}.$$

In case $\psi < \frac{\pi}{2}$, the harmonic function w is "not explicit" w.r.t. z. Here, we can to bound it **from below**:

Lemma

The harmonic function w satisfies the inequality

$$w(t) \geq \frac{2}{\pi} \left(\frac{\psi}{\sin \psi} \right)^{\frac{\pi}{2\psi}} \left(\frac{t}{\theta} \right)^{\frac{\pi}{2\psi}}, \qquad 0 < t \leq \theta.$$

Logarithmic stability

Theorem (Ait Ben Hassi-Chorfi-Maniar (2023))

Assume that $u_0 \in \mathcal{I}_{\varepsilon,M}$ *and the system is final state observable in* θ.
For $p \in \left(1, \frac{1}{1-\varepsilon}\right)$ *and* $s \in \left(0, 1 - \frac{1}{p}\right)$, $\exists K_1 > 0$:

$$\|u_0\|_H \leq K_1 \left(\frac{2\psi\Gamma\left(\frac{2\psi}{\pi}\right)}{\pi\left(-c_\psi p \log \|\mathbf{C}u\|_{L^2(0,\theta;Y)}\right)^{\frac{2\psi}{\pi}}} \right)^{\frac{s}{p}},$$

where $c_\psi = \frac{2}{\pi}\left(\frac{\psi}{\sin\psi}\right)^{\frac{\pi}{2\psi}}$, *provided that* $\|\mathbf{C}u\|_{L^2(0,\theta;Y)}$ *is sufficiently small.*

Ornstein–Uhlenbeck equation (Analytic case)

Consider the Ornstein-Uhlenbeck equation

$$\begin{cases} \partial_t y = \Delta y + Bx \cdot \nabla y, & 0 < t < \theta, \quad x \in \mathbb{R}^N, \\ y\big|_{t=0} = y_0, & \qquad\qquad\quad x \in \mathbb{R}^N. \end{cases}$$

B be a real constant $N \times N$-matrix.

KLIBANOV: logarithmic stability by **Carleman estimates** for **bounded coefficients** operators.

Ornstein–Uhlenbeck operator

- The operator $A := \Delta + Bx \cdot \nabla$, with its maximal domain, generates a (Ornstein-Uhlenbeck) C_0-semigroup on $L^2\left(\mathbb{R}^N\right)$.
- **Spectral condition:**

$$\sigma(B) \subset \mathbb{C}- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

  guarantees the existence of an **invariant measure** μ for the Ornstein-Uhlenbeck semigroup.

- The Ornstein-Uhlenbeck semigroup on $L^2_\mu := L^2\left(\mathbb{R}^N, \mathrm{d}\mu\right)$ is **analytic**.
- The analyticity angle ψ is such that $\psi < \dfrac{\pi}{2}$ in general.

References: R. Chill, E. Fašangová, G. Metafune, D. Pallara, A. Lunardi, ...

Observability inequality

Observation region: The observation operator: $\mathbf{C} = \mathbb{1}_\omega$

for an observation region $\omega \subset \mathbb{R}^N$:

$$\exists \delta, r > 0, \forall y \in \mathbb{R}^N, \exists y' \in \omega, \quad B\left(y', r\right) \subset \omega \text{ and } \left|y - y'\right| < \delta.$$

(L. Miller 2005, J. Le Rousseau & I. Moyano 2016).

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Example : open sets ω such that $\mathbb{R}^N \setminus \omega$ is bounded.

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Proposition (Beauchard-Pravda Strarov (2018))

$\exists \kappa_\theta = \kappa_\theta(\omega, \theta) > 0$:

$$\|y(\theta, \cdot)\|^2_{L^2_\mu} \leq \kappa_\theta \int_0^\theta \|y(t, \cdot)\|^2_{L^2_\mu(\omega)} \mathrm{d}t.$$

Stability estimate

Proposition

Let $p \in \left(1, \frac{1}{1-\varepsilon}\right)$ *and* $s \in \left(0, 1 - \frac{1}{p}\right)$. $\exists K_1 > 0$ *such that, for all* $\|y_0\|_{H^{2\varepsilon}_\mu} \leq M$*, we have*

$$\|y_0\|_{L^2_\mu} \leq K_1 \left(\frac{2\psi\Gamma\left(\frac{2\psi}{\pi}\right)}{\pi\left(-c_\psi p \log \|y\|_{L^2(0,\theta;L^2_\mu(\omega))}\right)^{\frac{2\psi}{\pi}}} \right)^{\frac{s}{p}},$$

provided that $\|y\|_{L^2(0,\theta;L^2_\mu(\omega))}$ *is sufficiently small.*

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E. M. Ait Ben Hassi, S. E. Chorfi and L. Maniar, Inverse problems for general parabolic systems and application to Ornstein-Uhlenbeck equation, *Discrete Contin. Dyn. Syst. - S* (2023), doi: 10.3934/dcdss.2022212.

Ornstein-Uhlenbeck equation in $L^2(\mathbb{R}^N)$ (non-analytic case)

Let $N \geq 1$ be an integer and let $\theta > 0$ be a fixed time.

We consider the Ornstein-Uhlenbeck equation given by

$$\begin{cases} \partial_t u = \Delta u + Bx \cdot \nabla u, & 0 < t < \theta, \quad x \in \mathbb{R}^N, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^N), \end{cases} \tag{4}$$

where B is a real constant $N \times N$-matrix **not necessarly** satisfaying

$$\sigma(B) \subset \mathbb{C}- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}.$$

Ornstein-Uhlenbeck semigroup in $L^2\left(\mathbb{R}^N\right)$

The Ornstein-Uhlenbeck C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ is given by Kolmogorov's formula

$$T(0) = I,$$

$$(T(t)f)(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_t}} \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{1}{4}\left\langle Q_t^{-1}y,y\right\rangle} f\left(\mathrm{e}^{tB}x - y\right)\mathrm{d}y, \quad t > 0, x \in \mathbb{R}^N$$

$$Q_t = \int_0^t \mathrm{e}^{sB}\,\mathrm{e}^{sB^*}\,\mathrm{d}s, \qquad t > 0.$$

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The C_0-semigroup $(T(t))_{t\geq 0}$ is **not analytic** in $L^2(\mathbb{R}^N)$.

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The C_0-semigroup $(T(t))_{t\geq 0}$ is **not analytic** in $L^2(\mathbb{R}^N)$.

Logarithmic convexity estimate

Proposition

There exists a constant $\kappa_\theta \geq 1$ *such that the following estimate holds*

$$\|T(t)f\| \leq \kappa_\theta \|f\|^{1-\frac{t}{\theta}} \|T(\theta)f\|^{\frac{t}{\theta}}, \qquad f \in L^2\left(\mathbb{R}^N\right), \qquad t \in [0,\theta]. \quad (5)$$

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Idea of the Proof.

$$T(t)f = S(t)\left(g_t * f\right),$$

where

$$g_t(y) = \frac{1}{\sqrt{(4\pi)^N \det Q_t}} e^{-\frac{1}{4}\left\langle Q_t^{-1}y,y\right\rangle}, \quad t > 0, \quad y \in \mathbb{R}^N.$$

and

$$\left(S(t)f\right)(x) = f\left(e^{tB}x\right), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N$$

$$\|S(t)f\| = e^{-\frac{t}{2}\mathrm{tr}(B)}\|f\|, \quad t \in \mathbb{R}.$$

Logarithmic estimate

We give a logarithmic estimate for a class of initial data.

Theorem

There exist positive constants C and C_1 depending on (N, θ, ω, R) such that, for all $u_0 \in \mathcal{I}_R$,

$$\|u_0\|_{L^2(\mathbb{R}^N)} \leq \frac{-C}{\log\left(C_1 \|u\|_{H^1(0,\theta;L^2(\omega))}\right)}$$

for $\|u\|_{H^1(0,\theta;L^2(\omega))}$ sufficiently small.

Time fractional equations

Let us consider the problem

$$\begin{cases} \partial_t^\alpha u(t) = Au(t), & t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{6}$$

where $A : D(A) \subset H \to H$ is a densely defined linear operator such that:

(i) A is self-adjoint,

(ii) A is bounded above: there exists $\kappa \geq 0$ such that $\langle Au, u\rangle \leq \kappa \|u\|^2$ for all $u \in D(A)$,

(iii) A has compact resolvent: there exists $\lambda > -\kappa$ such that the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ is compact.

Fractional Derivative

$$\partial_t^\alpha g(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t (t-s)^{-\alpha} \dfrac{\mathrm{d}}{\mathrm{d}s} g(s)\,\mathrm{d}s, & 0 < \alpha < 1, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} g(t), & \alpha = 1, \end{cases}$$

Logarithmic convexity

Theorem (Chorfi-Maniar-Yamamoto)

Let $0 < \alpha \leq 1$*. Let* u *be the solution to* (6)*. Then there exists a constant* $M \geq 1$ *such that*

$$\|u(t)\| \leq M \|u(0)\|^{1-\frac{t}{T}} \|u(T)\|^{\frac{t}{T}}, \qquad 0 \leq t \leq T. \tag{7}$$

Moreover, if $\kappa = 0$*, then we can choose* $M = 1$*.*

Ideas of the proof

- Use of the spectral representation

$$\|u(t)\|^2 = \sum_{n=1}^{\infty} \langle u_0, \varphi_n \rangle^2 \left(E_{\alpha,1}(-\lambda_n t^\alpha)\right)^2.$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}.$$

- The functions $t \mapsto (E_{\alpha,1}(-\lambda_n t^\alpha))^2$ are completely monotone on $[0, T]$ for $\lambda_n \geq 0$.

- Any completely monotone function $f : [0, \infty) \to (0, \infty)$ is log-convex.

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S. E. Chorfi, L. Maniar, M. Yamamoto, The backward problem for time fractional evolution equations, (2022) arXiv: 2211.16493.

Thank you for your attention