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Definitions and Notations

Consider the infinite-dimensional observation system

$$\begin{cases} \dot{z}(t) = A(t)z(t), \quad t \geq 0, \\ z(0) = x \in H \end{cases} \tag{1}$$

where

$$A(t) = A^0 - \alpha(t)BB^*, \quad t \geq 0. \tag{2}$$

- $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a positive, bounded and continuously differentiable function.
- $A_0 : D(A^0) \subset H \longrightarrow H$ is a skew-adjoint operator and H is a Hilbert space endowed with the norm $\| \ \|$.

- The control operator $B \in \mathcal{L}(U, H_{-1})$, which may be unbounded (i.e. not in $\mathcal{L}(U, H)$) , is acting on some Hilbert space U.
- Let H_1 be the Banach space $D(A^0)$ endowed with it's graph norm and $H_{-1} := D(A^0)'$ be the dual space of $D(A^0)$ with respect to the pivot space H.
- We have the inclusions $H_1 \subset H \subset H_{-1}$ with continuous embeddings.
- According to Stone's theorem , the operator A^0 generates a unitary group $(S(t))_{t\in\mathbb{R}}$.
- Then, the strongly continuous group $(S(t))_{t\in\mathbb{R}}$ can be extented to a strongly continuous unitary group $S_{-1}(t)$ on H_{-1} whose generator is the extension $A_{-1} \in \mathcal{L}(H, H_{-1})$ of $A^0 \in \mathcal{L}(H_1, H)$.

Definition 1

The operator $B \in \mathcal{L}(U, H_{-1})$ is called an admissible control operator (for S) if there exists $t_0 > 0$ such that

$$\int_0^{t_0} S_{-1}(t_0 - s)Bu(s)ds \in H, \quad \forall u \in L^2(0, t_0; U).$$

Proposition 1

Let $B \in \mathcal{L}(U, H_{-1})$ be admissible control operator. For $t > 0$, we define $\mathcal{B}_t \in \mathcal{L}\big(L^2(\mathbb{R}_+, U), H_{-1}\big)$ by

$$\mathcal{B}_t u = \int_0^t S_{-1}(t-r)Bu(r)dr.$$

Then, for every $t \geq 0$ we have

$$\mathcal{B}_t \in \mathcal{L}\big(L^2(\mathbb{R}_+, U), H\big).$$

Definition 2

Let $C \in \mathcal{L}(H_1, U)$. For $t \geq 0$, we define $\mathcal{C}_t \in \mathcal{L}\big(H_1, L^2([0,t], U)\big)$ by

$$\big(\mathcal{C}_t x\big)(s) = CS(s)x, \qquad \forall x \in D(A_0), \quad \forall s \in [0,t]. \tag{3}$$

The operator $C \in \mathcal{L}(H_1, U)$ is called an admissible observation operator (for S) if for some $t_0 > 0$, $\mathcal{C}_{t_0}$ has a continuous extension to H. Equivalently, $C \in \mathcal{L}(H_1, U)$ is an admissible observation operator if for some $t_0 > 0$ there exists $M_{t_0} > 0$ such that

$$\int_0^{t_0} \left\|CS(t)x\right\|_U^2 dt \leq M_{t_0} \|x\|_H^2, \quad \forall x \in D(A_0). \tag{4}$$

Proposition 2

If $C \in \mathcal{L}(H_1, U)$ is admissible. For $t \geq 0$, define

$$(\mathcal{C}_t x)(s) = CS(s)x, \qquad \forall x \in D(A_0), \quad \forall s \in [0, t].$$

Then, for every $t \geq 0$ we have

$$\mathcal{C}_t \in \mathcal{L}\big(H, L^2([0, t], U)\big).$$

Lemma 1

Suppose that $B \in \mathcal{L}(U, H_{-1})$. Then,
B is an admissible control operator (for S), if and only if, B^* is an admissible observation operator (for $S^* = -S$).

Definition 3

The system (A_0, B, C), with $(B, C) \in \mathcal{L}(U, H_{-1}) \times \mathcal{L}(H_1, U)$ is called compatible if for some $\lambda \in \rho(A^0)$ we have

$$R\big((\lambda - A_{-1})^{-1}B\big) \subset H_1. \tag{5}$$

Remark

It is obvious that if the inclusion (5) holds for some $\lambda \in \rho(A^0)$, then it holds for all $\lambda \in \rho(A^0)$ by the resolvent identity. Moreover, by the closed graph theorem, we have

$$C(\lambda - A_{-1})^{-1}B \in \mathcal{L}(U), \;\; \forall\, \lambda \in \rho(A^0).$$

- If not stated otherwise, in the sequel we always assume that the compatibility condition holds.

Definition 4

The pair $(B, C) \in \mathcal{L}(U, H_{-1}) \times \mathcal{L}(H_1, U)$ is called jointly admissible if B is an admissible control operator, C is an admissible observation operator, and there exists $t_0 > 0$ and $M > 0$ such that

$$\int_0^{t_0} \left\| C \int_0^r S_{-1}(r-s)Bu(s)ds \right\|_U^2 dr \leq M\|u\|^2_{L^2(0,t_0;U)} \tag{6}$$

for all $u \in W_0^{2,2}(0, t_0; U)$ where

$$W_0^{2,2}(0, t_0; U) := \left\{ u \in W^{2,2}(0, t_0; U) \middle| \quad u(0) = u'(0) = 0 \right\}.$$

Definition 5

Let A be the generator of a C_0-semigroup acting on a Banach space H, $B \in \mathcal{L}(U, H_{-1})$ and $C \in \mathcal{L}(H_1, U)$. The operator $BC \in \mathcal{L}(H_1, H_{-1})$ is called a Weiss-Staffans perturbation for A if the following conditions hold

- (i) $R\big((\lambda - A)^{-1}B\big) \subset H_1$ for some $\lambda \in \rho(A)$,
- (ii) B is an admissible control operator,
- (iii) C is an admissible observation operator,
- (iv) (B, C) is jointly admissible,
- (v) There exists $t_0 > 0$ such that $1 \in \rho(\mathcal{F}_{t_0}^{B,C})$ where

$$\mathcal{F}_{t_0}^{B,C} u(.) = C \int_0^{.} S_{-1}(. - s)Bu(s)ds, \quad \forall u \in W_0^{2,2}(0, t_0; U).$$

Theorem 1

Let $(A, D(A))$ be the generator of a C_0-semigroup $S(t)$ on a Banach space H, $B \in \mathcal{L}(U, H_{-1})$ and $C \in \mathcal{L}(H_1, U)$. Assume that $BC \in \mathcal{L}(H_1, H_{-1})$ is a Weiss-Staffans perturbation for A. Let $(A_{-1} + BC)_{|H}$ be the operator defined defined by

$$\left(A_{-1} + BC\right)_{|H} x = A_{-1} x + BCx$$

for every $x \in D\left(A_{-1} + BC\right)_{|H}$, wehere we have

$$D(A_{-1} + BC)_{|H} := \left\{x \in H_1 : \quad \left(A_{-1} + BC\right)_{|H} x \in H\right\}.$$

Theorem 1 continued

The operator $(A_{-1} + BC)_{|H}$ generates a C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ on H satisfying

$$T(t)x = S(t)x + \int_0^t S_{-1}(t-s)BCT(s)xds, \quad \forall t \geq 0, \forall x \in D(A_{-1}+BC)_{|H}.$$

Theorem 2

Let $A(t)$, $t \geq 0$ be as in (1.2) and assume that

(*i*) $B \in \mathcal{L}(U, H_{-1})$ is an admissible control operator.

(*ii*) The pair (B, B^*) is jointly admissible.

(*iii*) The system (A_0, B, B^*) is compatible.

(*iv*) There exists $t_0 > 0$ such that $\frac{1}{\sigma} \in \rho\big(\mathcal{F}_{t_0}\big)$ for all $\sigma \in \alpha\big([0, t_0]\big)$, where

$$\mathcal{F}_{t_0} u := \mathcal{F}_{t_0}^{B,B^*} u = B^* \int_0^{\cdot} S_{-1}(. - r)Bu(r)dr, \quad \forall u \in W_0^{2,2}(0, t_0; U).$$

(*v*) $D(A(t)) = D(A_0)$ for $t \geq 0$.

Theorem 2 continued

Then, the system governed by (1.1) and (1.2) is well posed.
More precisely, there exists an evolution family $(U(t,s))_{t\geq s\geq 0}$ consisting of contractions solving (1).
Hence, the solution of (1) is $z(t) = U(t,0)x$ for every $x \in H$.
Moreover, we have the mild solution

$$z(t) = S(t)x - \int_0^t S_{-1}(t-s)\alpha(s)BB^* z(s)ds, \quad t \geq 0, x \in D(A_{-1}+BB^*)_{|H}. \tag{7}$$

Lemma 2

Let $A(t)$, $t \geq 0$ be as in (1). Let $\tau > 0$. Then the operator defined on $D(A_{-1} + BB^*)_{|H}$ by

$$(\Lambda x)(t) = \sqrt{\alpha(t)}B^* z(t) = \sqrt{\alpha(t)}B^* U(t,0)x, \quad \forall x \in D(A_{-1} + BB^*)_{|H}$$

extended to a bounded operator $\Lambda \in \mathcal{L}\big(H, L^2(\mathbb{R}_+, U)\big)$ and

$$\int_0^\tau \|(\Lambda x)(t)\|_U^2 dt = \frac{\|x\|^2 - \|U(\tau,0)x\|^2}{2},$$

for all $x \in H$.

Proposition 3

Let $A(t)$, $t \geq 0$ be as in (2). Let $\tau > 0$ and z be the mild solution for the system (1). Then, there exists a positive constant a_τ such that for all $x \in D(A_{-1} + BB^*)_{|H}$

$$a_\tau \int_0^\tau \alpha(t) \|B^* S(t) x\|^2 dt \leq \int_0^\tau \alpha(t) \|B^* U(t,0) x\|^2 dt \leq \int_0^\tau \alpha(t) \|B^* S(t) x\|^2$$

Remark

Let $\tau > 0$. It may be useful to rewrite the last proposition as follows

$$a_\tau \int_0^\tau \|\Theta_\tau x(t)\|^2 dt \leq \int_0^\tau \|\Lambda_\tau x(t)\|^2 dt \leq \int_0^\tau \|\Theta_\tau x(t)\|^2 dt, \quad \forall x \in H,$$

where

$$\Theta_\tau x(t) = \sqrt{\alpha(t)}(\mathcal{C}_\tau x)(t), \quad \forall x \in H, \forall t \in (0, \tau),$$

$$(\Lambda_\tau x)(t) = (\Lambda x)(t), \quad \forall x \in H, \forall t \in (0, \tau)$$

and

$$(\mathcal{C}_\tau x)(t) = B^* S(t)x, \quad \forall x \in D(A), \forall t \in (0, \tau).$$

Definition 6

Let $\tau > 0$ and W be a subset of H.

- The pair (B^*, A) is approximately W-observable in time τ if $(\Lambda_\tau)_{|W}$ is injective which is equivalent to the condition

$$\forall x \in W, \quad \int_0^\tau \|(\Lambda_\tau x)(t)\|^2 dt = 0 \Longrightarrow x = 0.$$

- The pair (B^*, A) is exactly W-observable in time T if $(\Lambda_\tau)_{|W}$ is bounded from below, or equivalently

$$\exists C > 0, \forall x \in W, \quad \int_0^\tau \|\Lambda_\tau x(t)\|^2 dt \geq C\|x\|^2.$$

Definition 6 continued

- The pair (B^*, A^0) is approximately W-observable (resp. exactly W-observable) in time τ if $(\mathcal{C}_\tau)_{|W}$ is injective (resp. if $(\mathcal{C}_\tau)_{|W}$ is bounded from below).
- If $W = H$, we simply call the pair (B^*, A) (or the pair (B^*, A^0)) approximately observable or exactly observable in time τ.

Proposition 4

Let $A(t)$, $t \geq 0$, be as in (1). Suppose that $S(\tau) = I$ for some $\tau > 0$ and let $Z_\tau = U(\tau, 0)$. Then

$$N(I - Z_\tau) = N(\Theta_\tau) = \left\{ x \in H : \quad \int_0^\tau \|\Theta_\tau x(t)\|^2 dt = 0 \right\}.$$

• In particular, $\sigma(Z_\tau) \cap \mathbb{T} \subseteq \{1\}$.
• Moreover, if (B^*, A_0) is approximately observable in $(0, \tau)$, then $1 \notin \sigma_p(Z_\tau)$.
• Let $b_0 > 0$ and suppose that $\alpha \geq b_0$. If $1 \notin \sigma_p(Z_\tau)$, then (B^*, A_0) is approximately observable in $(0, \tau)$.

Proposition 5

Let $A(t)$, $t \geq 0$, be as in (1.2) and suppose that α is μ-periodic for some $\mu > 0$. Moreover, let $Z_\mu = U(\mu, 0)$ and W be a closed Z_μ-invariant subspace of H.

1. Assume that $r\big((Z_\mu)_{|W}\big) < 1$. Then
   - (*i*) the pair (B^*, A^0) is exactly W-observable in time $N\mu$ for some $N \in \mathbb{N}^*$.
   - (*ii*) if we suppose further that $S(\mu) = I$ then (B^*, A^0) is exactly W-observable in time μ.
2. Let $b_0 > 0$ and suppose that $\alpha \geq b_0$, then (B^*, A^0) is exactly W-observable in time $N\mu$ for some $N \in \mathbb{N}^*$, if and only if, $r\big((Z_\mu)_{|W}\big) < 1$.

Theorem 3

Let $A(t)$, $t \geq 0$, be as in (1) and suppose that α is μ-periodic for some $\mu > 0$. Moreover, let $Z_\mu = U(\mu, 0)$ and suppose that $S(\mu) = I$. Then

$$H = N(I - Z_\mu) \oplus \overline{R(I - Z_\mu)}.$$

Moreover, let Q be the projection onto $N(I - Z_\mu)$ along $\overline{R(I - Z_\mu)}$, then

(*i*) $\overline{R(I - Z_\mu)} = N(I - Z_\mu)^\perp$.

(*ii*) for any $x \in H$, $\|z(t) - z_0(t)\| \underset{t\to+\infty}{\longrightarrow} 0$ where z_0 is the μ-periodic solution of (1) such that $z_0(0) = Qx$.

Corollary

Let Let $A(t)$, $t \geq 0$, be as in (1.2) and suppose that α is μ-periodic for some $\mu > 0$. Moreover, let $Z_\mu = U(\mu, 0)$ and suppose that $S(\mu) = I$. The following results hold

- (i) if the pair (B^*, A^0) is approximately observable in time μ then the the system $(1, 2)$ is stable.
- (ii) Let $b_0 > 0$ and suppose that $\alpha \geq b_0$. Assume that the system $(1, 2)$ is stable. Then (B^*, A_0) is approximately observable in time μ.

Theorem 4

Let $A(t)$, $t \geq 0$, be as in (1) and suppose that α is μ-periodic for some $\mu > 0$. Moreover, let $Z_\mu = U(\mu, 0)$ and suppose that $S(\mu) = I$.

1. Suppose that
   - *(i)* there exists $b_0 > 0$ such that $\alpha \geq b_0$ and
   - *(ii)* the pair (B^*, A^0) is exactly observable in time μ.

   Then the system (1) is exponentially stable.
2. If the system (1) is exponentially stable then (B^*, A_0) is exactly observable in time μ.

We consider the following initial and boundary problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \alpha(t)\,\dfrac{\partial u}{\partial t}(\xi, t)\,\delta_\xi = 0,\ (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = u(1,t) = 0,\ t \in (0,+\infty), \\ u(x,0) = u^0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u^1(x),\ x \in (0,1), \end{cases} \tag{8}$$

where $\xi \in (0,1)$ and δ_ξ is the Dirac mass concentrated in the point $\xi \in (0,1)$.

In this case, we have

- $H = H_0^1(0,1) \times L^2(0,1)$, $H_1 = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1)$.
- $H_{-1} = L^2(0,1) \times H^{-1}(0,1)$.

- $A^0 = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix} : H_1 \subset H \to H.$
- $Bk = \begin{pmatrix} 0 \\ k\delta_\xi \end{pmatrix}, \forall k \in \mathbb{C}$ and $B^* \begin{pmatrix} f \\ g \end{pmatrix} = g(\xi), \forall (f, g) \in H_1.$
- $BB^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ g(\xi)\, \delta_\xi \end{pmatrix}, \forall (f, g) \in H_1.$

Remark

For any $\xi \in (0, 1)$ the system described by (8) with $\alpha \equiv 1$, is not exponentially stable in H, see [6.].

Lemma 3 (see [6.])

Suppose that $(u_0, u_1) \in H$. Then the initial and boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \ (x,t) \in (0,1) \times (0,+\infty), \\ u(0,t) = u(1,t) = 0, \ t \in (0,+\infty), \\ u(x,0) = \varphi_0(x), \ \frac{\partial u}{\partial t}(x,0) = \varphi_1(x), \ x \in (0,1), \end{cases} \tag{9}$$

admit a unique solution $\varphi \in C(0,T;H_0^1) \cap C^1(0,T;L^2(0,1))$ and there exists a constant $C > 0$ such that

$$\int_0^2 |\frac{\partial \varphi}{\partial t}(\xi,t)|^2 dt \le C \| \begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} \|_H^2. \tag{10}$$

Lemma 4 (see [6.])

Let $u \in L^2(0, T)$ $(T > 0)$, then the problem

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial t^2} - \dfrac{\partial^2 \psi}{\partial x^2} + u(t)\,\delta_\xi = 0,\ (x, t) \in (0, 1) \times (0, +\infty), \\ \psi(0, t) = \psi(1, t) = 0,\ t \in (0, +\infty), \\ \psi(x, 0) = 0,\ \dfrac{\partial \psi}{\partial t}(x, 0) = 0,\ x \in (0, 1), \end{cases} \tag{11}$$

admit a unique solution $\psi \in C(0, T; H_0^1(0, 1)) \cap C^1(0, T; L^2(0, 1))$ and there exists a constant $M > 0$ such that

$$\int_0^2 |\frac{\partial \psi}{\partial t}(\xi, t)|^2 dt \leq M \|u\|_{L^2(0,2;\mathbb{C})}. \tag{12}$$

Theorem 5

Let $M > 0$ defined as in Lemma 4. Suppose that α is continuously differentiable, 2-periodic and $\alpha \leq \frac{1}{M}$ then, for any $\xi \in (0,1)$, The initial boundary problem (8) is well-posed and not exponentially stable in $H_0^1 \times L^2(0,1)$.

Sketch of the proof

i. Let $(\varphi_0, \varphi_1) \in H = H_0^1(0,1) \times L^2(0,1)$. Then, $S(t)\begin{pmatrix}\varphi_1 \\ \varphi_0\end{pmatrix} = \tilde{\varphi} := \begin{pmatrix}\varphi \\ \frac{\partial \varphi}{\partial t}\end{pmatrix}$, where φ is the solution of (9). Note that $\tilde{\varphi}(t)$ is 2-periodic.

ii. By using Lemma 3, we get

$$\int_0^2 \left| B^* S(t)\begin{pmatrix}\varphi_1 \\ \varphi_0\end{pmatrix}\right|^2 dt = \int_0^2 \left|\frac{\partial \varphi}{\partial t}(\xi, t)\right|^2 dt \leq C \| \begin{pmatrix}\varphi_1 \\ \varphi_0\end{pmatrix} \|_H^2.$$

Then assertion (i) in Theorem 2 is satisfied.

Sketch of the proof continued

i. Let $u \in W_0^{2,2}(0,2;\mathbb{C})$ and $P(t) = \int_0^t S_{-1}(t-s)Bu(s)ds$. Then P is solution of

$$\begin{cases} P'(t) = A^0 P(t) + Bu(t), \quad t \in (0,+\infty), \\ P(0) = 0, \end{cases} \tag{13}$$

Putting $P = \begin{pmatrix} \psi \\ \nu \end{pmatrix}$, (13) is equivalent to (11). Then, by Lemma 4,

$$\int_0^2 |B^* P(t)|^2 dt = \int_0^2 |\frac{\partial \psi}{\partial t}(\xi,t)|^2 dt \leq M \|u\|_{L^2(0,2;\mathbb{C})}.$$

Hence, assertion (*ii*) in Theorem 2 is satisfied.

Sketch of the proof continued

ii. Moreover, again from [6.], we have that for every $\lambda \in \mathbb{C}$ such that $Re(\lambda) > 0$,

$$R((\lambda - A^0)^{-1}B) \subset H_1.$$

So the assertions (*iii*) of Theorem 2.2 are satisfied.

iii. Since $\|\alpha\|_\infty < \dfrac{1}{M}$, we deduce using (12),

$$\frac{1}{\sigma} \in \rho(\mathcal{F}_2), \quad \forall\, \sigma \in \alpha([0,2]).$$

Then, assertion (*iv*) in Theorem 2 is satisfied.

Sketch of the proof continued

iv. By using continuous fractions, there exists a sequence q_m of positive integer numbers such that $q_m \to \infty$ and

$$|\sin(q_m \pi \xi)| \leq \frac{\pi}{q_m}, \ \ \forall\, m \geq 1.$$

v. Then we take $(\varphi_0, \varphi_1) = \big(0, \sin(q_m \pi x)\big)$ in (9) to conclude that (B^*, A^0) is not exactly observable in time 2. Theorem 4 achieves the proof.

Theorem 6

Suppose that α satisfies the hypothesis in Theorem 5 and $\alpha \geq b_0$ for some positive number b_0. Then, the system described by (8) is strongly stable if and only if $\xi \notin \mathbb{Q}$.

Sketch of the proof

- Let $(\varphi_0, \varphi_1) \in H_1$. We have
  $\varphi_0(x) = \sum_{k=1}^{+\infty} a_k \sin(k\pi x)$ and $\varphi_1(x) = \sum_{k=1}^{+\infty} b_k \sin(k\pi x)$ where
  $(ka_k), (b_k) \in l^2$.
- Then the solution φ of (9) is given by

$$\varphi(x,t) = \sum_{k\geq 1} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x).$$

- Recall that $B^* S(t) \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \frac{\partial \varphi}{\partial t}(\xi, t)$.

Sketch of the proof continued

- By using Ingham inequality, we get,

$$\int_0^2 |\alpha(t)| \left| B^* S(t) \begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} \right|^2 dt = \int_0^2 |\alpha(t)| \left| \frac{\partial \varphi}{\partial t}(\xi, t) \right|^2 dt$$

$$\geq c \sum_{k=1}^{+\infty} \frac{1}{k^2} \left(k^2 |a_k|^2 + |b_k|^2 \right) \sin^2(k\pi\xi)$$

for some positive constant c. This implies the approximate observability of (B^*, A^0) in time 2 for $\xi \notin \mathbb{Q}$.

Sketch of the proof continued

- For the converse, suppose that $\xi = \dfrac{p}{q}$ with $(p, q \in \mathbb{N})$, $q \neq 0$. It suffices to take $(\varphi_0, \varphi_1) = (0, \sin(qx))$ to conclude that (B^*, A) is not approximate observable (in time 2).

We consider the following initial and boundary problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \alpha(t)\, \dfrac{\partial u}{\partial t}(\xi, t)\, \delta_\xi = 0,\ (x, t) \in (0, 1) \times (0, +\infty), \\ u(0, t) = \dfrac{\partial u}{\partial t}(1, t) = 0,\ t \in (0, +\infty), \\ u(x, 0) = u^0(x),\ \dfrac{\partial u}{\partial t}(x, 0) = u^1(x),\ x \in (0, 1), \end{cases} \tag{14}$$

where $\xi \in (0, 1)$ and δ_ξ is the Dirac mass concentrated in the point $\xi \in (0, 1)$. In this case we have

- $H = V \times L^2(0, 1)$ where $V := \{u \in H^1(0, 1) \mid u(0) = 0\}$.
- $H_1 = \{(u, v) \in H^2(0, 1) \times H^1(0, 1),\ u(0) = v(0) = \frac{du}{dx}(1) = 0\}$.
- $H_{-1} = L^2(0, 1) \times H^{-1}(0, 1)$.

- $A^0 = \begin{pmatrix} 0 & I \\ \frac{d^2}{dx^2} & 0 \end{pmatrix} : H_1 \subset H \to H.$
- $Bk = \begin{pmatrix} 0 \\ k\delta_\xi \end{pmatrix}, \forall k \in \mathbb{C}.$
- $B^* \begin{pmatrix} f \\ g \end{pmatrix} = g(\xi), \forall (f, g) \in H_1.$
- $BB^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ g(\xi)\, \delta_\xi \end{pmatrix}, \forall (f, g) \in H_1.$

Remark

The case of $\alpha \equiv 1$ was studied in [3.], see also [4.], The authors proved that the system is exponentially stable in the energy space H if and only if $\xi \in (0, 1)$ admits a coprime factorization

$$\xi = \frac{p}{q} \quad \text{with } p \text{ odd.} \tag{15}$$

The fastest decay rate is obtained if $\xi = \dfrac{1}{2}$. They proved also that the strong stability, with initial data in H, is obtained if and only if

$$\xi \neq \frac{2p}{2q-1} \quad \forall p, q \in \mathbb{N}.$$

Theorem 7

Suppose that α is continuously differentiable, 2-periodic. There exists $M > 0$ such that, if $\alpha \leq M$ then,
(i) For any $\xi \in (0, 1)$, The initial boundary problem (14) is well-posed in $H = V \times L^2(0, 1)$.
If in addition, there exists $b_0 > 0$ such that $\alpha \geq b_0$, we have
(ii) The initial boundary problem (14) is strong stability, with initial data in H, if and only if $\xi \neq \dfrac{2p}{2q-1}$ for all $p, q \in \mathbb{N}$.
(iii) The initial boundary problem (14), with initial data in H is exponentially stable in the energy space H if and only if $\xi \in (0, 1)$ admits a coprime factorization $\xi = \dfrac{p}{q}$ with p odd.

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Thank for your attention