Compactness of the Resolvent for Kramers-Fokker-Planck Operators with Polynomial Potential

Mona Ben Said

University of Monastir

Monastir, May-08-2023

Part I: Context and objectives

- In this presentation, we consider the Kramers-Fokker-Planck operator given by

$$K_V = \underbrace{\Big(p \cdot \partial_q - \partial_q V(q) \cdot \partial_p\Big)}_{:=X_V} + \underbrace{\frac{1}{2}\Big(-\Delta_p + p^2\Big)}_{:=O_p}, \tag{2.1}$$

where $(q, p) \in \mathbb{R}^{2d}$ and $V(q)$ is a real-valued potential.

- This presentation is concerned with the study of some spectral properties and compactness criteria for the resolvent of the Kramers-Fokker-Planck operator K_V with a polynomial potential $V(q)$.

▶ This spectral study is intimately related to the study of the stochastic non-reversible diffusion processes of the statistical physics:

From the Langevin equation (1908 - Paul Langevin)

$$\begin{cases} dq = p\,dt \\ ma\,dt = dp = \underbrace{-\nabla V(q)\,dt}_{\text{derived from a potential}} \quad \underbrace{-\gamma p\,dt}_{\text{friction force}} + \underbrace{\sqrt{2m\gamma}\,dW_t}_{\text{random force}}, \end{cases}$$

we get the Fokker-Planck equation

$$\begin{cases} \partial_t u(t,q,p) + K_V u(t,q,p) = 0 \\ u(0,q,p) = u_0(q,p)\ . \end{cases} \tag{2.2}$$

Elementary properties of K_V:

$$K_V = \underbrace{\Big(p\cdot\partial_q - \partial_q V(q)\cdot\partial_p\Big)}_{:=X_V} + \underbrace{\frac{1}{2}\Big(-\Delta_p + p^2\Big)}_{:=O_p}$$

The Kramers-Fokker-Plank operator $(K_V, \mathcal{C}_0^\infty(\mathbb{R}^{2d}))$.

- is neither elliptic nor self-adjoint.
- is essentially maximal accretive. Therefore, the domain of the closure of K_V is given by

$$D(K_V) = \Big\{u \in L^2(\mathbb{R}^{2d}) : \ K_V u \in L^2(\mathbb{R}^{2d})\Big\}$$

and $-K_V$ is a generator of the contraction semi-group $\Big\{e^{-tK_V}\Big\}_{t\geq 0}$.

Link with the Witten Laplacians $\Delta_V^{(0)}$: [HeNi]

The Witten Laplacian $(\Delta_V^{(0)}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ is defined by

$$\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta V(q) \ . \tag{2.3}$$

Conjecture 2.1 (Helffer-Nier [HeNi])

For $V \in \mathcal{C}^\infty(\mathbb{R}^d)$,

$$K_V \text{ has a compact resolvent} \Leftrightarrow \Delta_V^{(0)} \text{ has a compact resolvent.}$$

[HeNi] B. Helffer, F. Nier: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, 1862. Springer-Verlag. x+209 pp, (2005)

Assumption 2.1

- *The potential V is a $\mathcal{C}^\infty$ function and there exist $n \geq 1$ and, for all $\alpha \in \mathbb{N}^d$, a positive constant C_α so that*

$$\forall q \in \mathbb{R}^d, \quad |\partial_q^\alpha V(q)| \leq C_\alpha \left(1 + \langle q \rangle^{2n - \min\{|\alpha|, 2\}}\right).$$

- *There exists two constants $C_0 = C_0(V) > 0$ and $C_1 = C_1(V) > 0$ such that*

$$\forall q \in \mathbb{R}^d, \quad \pm V(q) \geq C_0^{-1} \langle q \rangle^{2n} - C_0 \quad \text{and} \quad |\partial_q V(q)| \geq C_1^{-1} \langle q \rangle^{2n-1} - C_1 \,,$$

Theorem 2.1 (Hérau-Nier [HerNi])

If $V(q)$ satisfies Assumption 2.1 then there is a constant $c_V > 0$ such that,

$$\|\Lambda^\epsilon u\|^2_{L^2(\mathbb{R}^{2d})} \leq C_V \left(\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})}\right), \qquad \forall u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \tag{2.4}$$

$$\epsilon = \min\left(\frac{1}{4}, \frac{1}{4n-2}\right), \qquad \Lambda = \left(1 - \Delta_p - \Delta_q + |\partial_q V(q)|^2 - \Delta V(q) + \frac{1}{2}|p|^2\right)^{\frac{1}{2}}.$$

[HerNi] F. Hérau, F. Nier: Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. Arch. Ration. Mech. Anal. 171, no. 2, 151-–218, (2004).

Known Results:

Known Case:

$$\exists C > 0, \qquad |\mathrm{Hess}\, V(q)| \leq C \langle \partial_q V(q) \rangle^s, \qquad \text{with} \quad s \leq \frac{4}{3}\,,$$

where here and throughout this presentation we use the notation $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$.

Interesting case: Degenerate potential at infinity:

Example: $V(q_1, q_2) = q_1^2 q_2^2$

$\Delta^{(0)}_{-V}$ has a compact resolvent.

$\Delta^{(0)}_{+V}$ has not a compact resolvent.

[HeNi] B. Helffer, F. Nier: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, 1862. Springer-Verlag. x+209 pp, (2005)

Notations 2.1

For $r \in \mathbb{N}$, we denote E_r the set of polynômials with degrée less than or equal to r:

$$E_r = \{P \in \mathbb{R}[q_1, q_2, \cdots, q_d]\,, \quad \deg P \le r\}\,.$$

For a polynomial $V(q) \in E_r$, we define the function $R_V^{\geq 1} : \mathbb{R}^d \to \mathbb{R}$ by

$$R_V^{\geq 1}(q) = \sum_{1 \le |\alpha| \le r} \left|\partial_q^\alpha V(q)\right|^{\frac{1}{|\alpha|}}\,. \tag{2.5}$$

Definition 2.1

A set $\mathcal{L}$ in E_r satisfying the following three conditions is called canonical set.

1. *If $P \in \mathcal{L}$ and $y \in \mathbb{R}^d$, then the polynôme defined by*

$$Q(q) = P(q+y) - P(y) , \quad \forall q \in \mathbb{R}^d ,$$

   is also in $\mathcal{L}$.

2. *If $P \in \mathcal{L}$ and $\lambda > 0$ then $Q(q) = P(\lambda q) \in \mathcal{L}$.*
3. *$\mathcal{L}$ is a closed subset of E_r.*

Notation 2.1

For a polynomial $V \in E_r$, we denote by $\mathcal{L}_V$ the smallest canonical closed set containing V.

[HeNi] B. Helffer, F. Nier: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, 1862. Springer-Verlag. x+209 pp, (2005)

Theorem 2.2 (Helffer-Nourrigat Theorem $\rightarrow$ Helffer-Nier)

For a potential $V \in E_r$, we suppose that:

1. $\lim_{|q| \to +\infty} R_V^{\geq 1}(q) = +\infty$,
2. *The canonical set $\mathcal{L}_V \cap E_{r-1}$ does not contain any non zero polynomial having a local minimum.*

Then the Witten Laplacian $\Delta_V^{(0)}$ has compact resolvent.

B. Helffer, F. Nier: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, 1862. Springer-Verlag. x+209 pp, (2005)

Known Results: W-Xi-Li Results

Notation 2.2

Let $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$*. Denote by* $\lambda_l(q)$, $1 \leq l \leq d$*, the eigenvalues of the Hessian matrix*

$$(\partial_{q_i q_j} V(q))_{1 \leq i,j \leq d} \ .$$

With each $q \in \mathbb{R}^d$*, we associate a set* I_q *of indexes defined by*

$$I_q = \{1 \leq l \leq d \ , \ \text{such that } \lambda_l(q) > 0\} \ .$$

[Li] W.-X. Li: Compactness criteria for the resolvent of Fokker-Planck operator.prepublication. ArXiv1510.01567, (2015).

Known Results: W-Xi-Li Results

Theorem 2.3 (W-Xi-Li [Li])

Let $V(q) \in \mathcal{C}^2(\mathbb{R}^d)$. Suppose that there exists a constant $c > 0$ such that,

$$\forall\, q \in \mathbb{R}^d \;, \qquad \sum_{j \in I_q} \lambda_j(q) \leq c \langle \partial_q V(q) \rangle^{\frac{4}{3}} \;,$$

Then the following conclusions hold.
(i) There exists a constant $c > 0$ such that for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\| |\partial_q V(q)|^{\frac{1}{16}} u \|_{L^2(\mathbb{R}^{2d})} \leq c \left(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})} \right) . \tag{2.6}$$

(ii) If we suppose futhermore the existence of a constant $\alpha \geq 0$ such that

$$\lim_{|q| \to +\infty} (\alpha |\partial_q V(q)|^2 - \Delta_q V(q)) = +\infty \;,$$

then there exists a constant $\widetilde{c}_\alpha > 0$ such that for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$,

$$\| |\alpha |\partial_q V(q)|^2 - \Delta_q V(q)|^{\frac{1}{80}} u \|_{L^2(\mathbb{R}^{2d})} \leq \widetilde{c}_\alpha \left(\|K_V u\|_{L^2(\mathbb{R}^{2d})} + \|u\|_{L^2(\mathbb{R}^{2d})} \right) . \tag{2.7}$$

Part II: Case of a polynomial potential with degree $r \leq 2$

Case of a polynomial potential with degree $r \leq 2$

Notations 4.1

- *For an arbitrary polynomial $V(q)$ of degree r, we denote for all $q \in \mathbb{R}^d$*

$$Tr_{+,V}(q) = \sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\, V(q)) \\ \nu > 0}} \nu(q)\,, \qquad Tr_{-,V}(q) = -\sum_{\substack{\nu \in \mathrm{Spec}(\mathrm{Hess}\, V(q)) \\ \nu \leq 0}} \nu(q)\ .$$

- *For a polynomial $V(q)$ of degree $r \leq 2$, we denote*

$$A_V = \max\{(1 + Tr_{+,V})^{2/3}, 1 + Tr_{-,V}\}$$

$$B_V = \max\{\min_{q \in \mathbb{R}^d} |\nabla V(q)|^{4/3}, \frac{1 + Tr_{-,V}}{\log(2 + Tr_{-,V})^2}\}\ .$$

M. Ben Said, F. Nier, J. Viola : Quaternionic structure and analysis of some Kramers-Fokker-Planck operators. Asymptotic Analysis. 2020;119(1-2):87-116.

Case of a polynomial potential with degree $r \leq 2$

Theorem 4.1 (Ben Said, Nier, Viola [BNV])

Let $V(q)$ be a potential with degree $r \leq 2$. Then there exists a constant $c > 0$ that does not depend on V such that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + A_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q)\rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q\rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})} \Big) \quad (4.1)$$

holds for all $u \in D(K_V)$.

M. Ben Said, F. Nier, J. Viola : Quaternionic structure and analysis of some Kramers-Fokker-Planck operators. Asymptotic Analysis. 2020;119(1-2):87-116.

Case of a polynomial potential with degree $r \leq 2$

Theorem 4.2 (Ben Said, Nier, Viola [BNV])

Let $V(q)$ a polynomial with degree $r \leq 2$. Then there is a constant $c > 0$, independent of the polynomial V, so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq c\, B_V \|u\|^2_{L^2(\mathbb{R}^{2d})} \,, \tag{4.2}$$

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{c}{1 + \frac{A_V}{B_V}} \Big(\|O_p u\|^2_{L^2(\mathbb{R}^{2d})} + \|X_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle \partial_q V(q) \rangle^{2/3} u\|^2_{L^2(\mathbb{R}^{2d})} + \|\langle D_q \rangle^{2/3} u\|_{L^2(\mathbb{R}^{2d})} \Big) \,. \tag{4.3}$$

hold for all $u \in D(K_V)$.

M. Ben Said, F. Nier, J. Viola : Quaternionic structure and analysis of some Kramers-Fokker-Planck operators. Asymptotic Analysis. 2020;119(1-2):87-116.

Case of a polynomial potential with degree $r \leq 2$

Lemma 4.1 (Ben Said, Nier, Viola [BNV])

Let $V(q) = -\frac{\nu q^2}{2}$ *where* $q \in \mathbb{R}$ *and* $\nu > 0$. *For every* $t \geq 0$, *one has*

$$\|e^{-tK_V}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} = e^{-\operatorname{Argsh}\left(\frac{\operatorname{sh}(\frac{t\sqrt{4\nu+1}}{2})}{\sqrt{4\nu+1}}\right)}. \tag{4.4}$$

Therefore, there is a constant $c > 0$ *such that, for all* $\nu > c$,

$$\|K_V^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} := \|\int_0^{+\infty} e^{-tK_V}\,dt\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq c\frac{\log(\nu)}{\sqrt{\nu}}\ .$$

Proposition 4.1 (Ben Said, Nier, Viola [BNV])

Let $V(q) = -\frac{\nu q^2}{2}$ *where* $q \in \mathbb{R}$ *and* $\nu >> 1$. *There is a function* $u \in L^2(\mathbb{R}^2)$ *such that*

$$\|K_V u\|_{L^2(\mathbb{R}^2)} \leq c\ \frac{\sqrt{\nu}}{\sqrt{\log(\nu)}}\|u\|_{L^2(\mathbb{R}^2)}$$

where $c > 0$ *is a constant that does not depend on the parameter* $\nu \gg 1$.

Part III: Case of a polynomial potential with degree $r \geq 3$

Notation 5.1

Given a polynomial $V(q)$ with degree $r \geq 3$ we define

$$R_V^{\geq 3}(q) = \sum_{3 \leq |\alpha| \leq r} \left| \partial_q^\alpha V(q) \right|^{\frac{1}{|\alpha|}} , \tag{5.1}$$

Case of a polynomial potential with degree $r \geq 3$

Notation 5.2

For $\kappa > 0$, we set

$$\Sigma(\kappa) = \left\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq \kappa \Big(|\mathrm{Hess}\ V(q)| + R_V^{\geq 3}(q)^4 + 1 \Big) \right\} .$$

Assumption 5.1

There exist large constants $\kappa_0, C_1 > 1$ such that for all $\kappa \geq \kappa_0$ the polynomial $V(q)$ satisfies the following properties

$$\mathrm{Tr}_{-,V}(q) > \frac{1}{C_1} \mathrm{Tr}_{+,V}(q), \ \ \textit{for all}\ \ q \in \mathbb{R}^d \setminus \Sigma(\kappa)\ \textit{with}\ |q| \geq C_1 \ , \tag{5.2}$$

moreover if $\mathbb{R}^d \setminus \Sigma(\kappa)$ is not bounded

$$\lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^d \setminus \Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\ V(q)|} = 0 \ . \tag{5.3}$$

Examples:

Example 1: $V(q_1, q_2) = -q_1^2 q_2^2$, with $q = (q_1, q_2) \in \mathbb{R}^2$,

- By direct computation

$$\partial_q V(q) = \begin{pmatrix} -2q_1 q_2^2 \\ -2q_2 q_1^2 \end{pmatrix} \ , \ |\partial_q V(q)| = 2|q_1 q_2||q| \ ,$$

$$\mathrm{Hess}\ V(q) = \begin{pmatrix} -2q_2^2 & -4q_1 q_2 \\ -4q_1 q_2 & -2q_1^2 \end{pmatrix} \ , \ |\mathrm{Hess}\ V(q)| = 2\sqrt{|q|^4 + 6q_1^2 q_2^2} \asymp |q|^2 \ ,$$

$$R_V^{\geq 3}(q) = |4q_2|^{1/3} + |4q_1|^{1/3} + 2 \times 4^{1/4} \ .$$

- $\mathit{Trace}(\mathrm{Hess}\ V(q)) = -2|q|^2$, hence

$$\mathrm{Tr}_{-,V}(q) > \mathrm{Tr}_{+,V}(q) \quad \text{for all } q \in \mathbb{R}^2 \setminus \{0\} \ .$$

- Furthermore for $\kappa > 1$

$$\lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^2 \setminus \Sigma(\kappa)}} \frac{R_V^{\geq 3}(q)^4}{|\mathrm{Hess}\ V(q)|} = \lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^2 \setminus \Sigma(\kappa)}} \frac{|q|^{4/3}}{|q|^2} = 0 \ .$$

Below we sketch in a blue color

$$\Sigma(800) = \left\{ q \in \mathbb{R}^d,\ |\nabla V(q)|^{\frac{4}{3}} \geq 800 \left(|\mathrm{Hess}\ V(q)| + R_V^{\geq 3}(q)^4 + 1 \right) \right\}$$

.

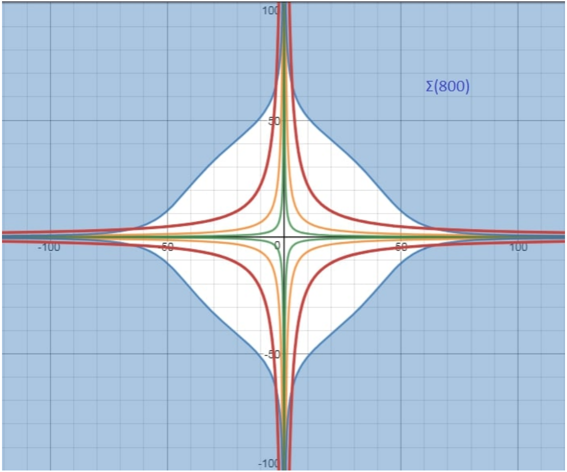

Figure: Contour lines of $V(q_1, q_2) = -q_1^2 q_2^2$

Case of a polynomial potential with degree $r \geq 3$

Example 2: $V(q) = -q_1^2(q_1^2 + q_2^2), \qquad q = (q_1, q_2) \in \mathbb{R}^2$,

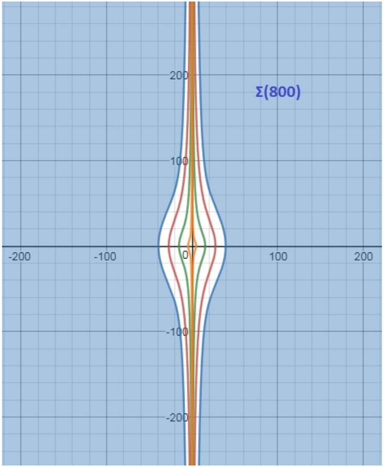

Figure: Contour lines of $V(q_1, q_2) = -q_1^2(q_1^2 + q_2^2)$

Example 3: For $\epsilon \in \mathbb{R} \setminus \{0, -1\}$, we consider $V(q_1, q_2) = (q_1^2 - q_2)^2 + \epsilon q_2^2$.

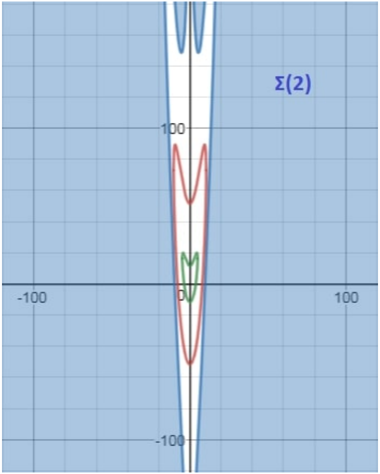

Figure: Contour lines of $V(q_1, q_2) = (q_1^2 - q_2)^2 + 0.5 q_2^2$.

Case of a polynomial potential with degree $r \geq 3$

Theorem 5.1 (Ben Said [Ben])

Let $V(q)$ be a polynomial verifying Assumption 5.1. Then there exists a strictly positive constant $C_V > 1$ (depending on V) such that

$$\|K_V u\|_{L^2}^2 + C_V\|u\|_{L^2}^2 \geq \frac{1}{C_V}\Big(\|L(O_p)u\|_{L^2}^2 + \|L(\langle|\nabla V(q)|\rangle^{\frac{2}{3}})u\|_{L^2}^2 + \|L(\langle|\mathrm{Hess}\ V(q)|\rangle^{\frac{1}{2}})u\|_{L^2}^2 + \|L(\langle|D_q|\rangle^{\frac{2}{3}})u\|_{L^2}^2\Big)\ , \tag{5.4}$$

holds for all $u \in D(K_V)$ where $L(s) = \frac{s+1}{\log(s+1)}$ for any $s \geq 1$.

Corollary 5.1 (Ben Said [Ben])

If $V(q)$ is a polynomial that satisfies Assumption 5.1, then the Kramers-Fokker-Planck operator K_V has a compact resolvent.

M. Ben Said: Global subelliptic estimates for Kramers-Fokker-Planck operators with some class of polynomials. J Inst Math Jussieu. 2020:1-37. https://doi.org/10.1017/S1474748020000249.

Sketch of the Proof of Theorem 5.1:

Lemma 5.1

Given a polynomial $V(q)$ with degree $r \geq 3$, there exists a locally finite partition of unity

$$\sum_{j\in\mathbb{N}} \chi_j^2(q) = \sum_{j\in\mathbb{N}} \widetilde{\chi}_j^2\left(R_V^{\geq 3}(q_j)(q-q_j)\right) = 1, \tag{5.5}$$

where

$$\operatorname{supp} \widetilde{\chi}_j \subset B(0,a) \quad \textit{and} \quad \widetilde{\chi}_j \equiv 1 \textit{ in } \quad B(0,b)$$

for some $q_j \in \mathbb{R}^d$ with $0 < b < a$ independent of $j \in \mathbb{N}$.

Lemma 5.2

Assume $V \in \mathbb{R}[q_1, \ldots, q_d]$ with degree $r \in \mathbb{N}$. For a locally finite partition of unity namely $\sum_{j \in \mathbb{N}} \chi_j^2(q) = 1$ one has

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} = \sum_{j \in \mathbb{N}} \left(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(p\partial_q \chi_j)u\|^2_{L^2(\mathbb{R}^{2d})} \right) , \tag{5.6}$$

for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

In particular, when the degree of V is larger than two and the cutoff functions χ_j have the form (5.5), there exists a constant $c_{d,r} > 0$ (depending only on the dimension d and the degree of V) so that

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j \in \mathbb{N}} \left(\|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c_{d,r} R_V^{\geq 3}(q_j)^2 \|p\chi_j u\|^2_{L^2(\mathbb{R}^{2d})} \right) , \tag{5.7}$$

holds for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$.

Notations 5.1

Let V be a polynomial of degree $r \geq 3$. Consider a locally finite partition of unity $\sum_{j\in\mathbb{N}} \chi_j^2(q) = 1$ described as in (5.5).

For a given $\kappa > 0$ and all indices $j \in \mathbb{N}$, let $V_j^{(2)}$ be the polynomial of degree less than three given by

$$V_j^{(2)}(q) = \sum_{0\leq|\alpha|\leq 2} \frac{\partial_q^\alpha V(q'_j)}{\alpha!}(q - q'_j)^\alpha \ , \tag{5.8}$$

where

$$\begin{cases} q'_j = q_j & \textit{if} \quad \mathrm{supp}\ \chi_j \subset \Sigma(\kappa) \\ q'_j \in (\mathrm{supp}\ \chi_j) \cap \left(\mathbb{R}^d \setminus \Sigma(\kappa)\right) & \textit{otherwise.} \end{cases}$$

We associate with each polynomial $V_j^{(2)}$ the Kramers-Fokker-Planck operator $K_{V_j^{(2)}}$. Observe that using the parallelogram law $2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 = \|x-y\|^2 \geq 0$,

$$\begin{aligned}\sum_{j\in\mathbb{N}} \|K_V(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} &= \sum_{j\in\mathbb{N}} \|K_{V_j^{(2)}}(\chi_j u) + (K_V - K_{V_j^{(2)}})(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} \\ &\geq \sum_{j\in\mathbb{N}} \left(\frac{1}{2}\|K_{V_j^{(2)}}(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - \|(\nabla V(q) - \nabla V_j^{(2)}(q))\partial_p(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})}\right) \\ &\geq \sum_{j\in\mathbb{N}} \left(\frac{1}{2}\|K_{V_j^{(2)}}(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c'_{d,r} R_V^{\geq 3}(q_j)^2 \|\partial_p \chi_j u\|^2_{L^2(\mathbb{R}^{2d})}\right) . \end{aligned} \tag{5.9}$$

Combining (5.7) and (5.9) we get immediately

$$\|K_V u\|^2_{L^2(\mathbb{R}^{2d})} + \|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{j\in\mathbb{N}} \left(\frac{1}{2}\|K_{V_j^{(2)}}(\chi_j u)\|^2_{L^2(\mathbb{R}^{2d})} - c"_{d,r} R_V^{\geq 3}(q_j)^4 \|\chi_j u\|^2_{L^2(\mathbb{R}^{2d})}\right) . \tag{5.10}$$

Part IV: Perspectives

Perspectives and some Open Questions:

1. Is it possible to extend the result of Theorem 5.1 in the case of a degenerate non polynomial potential?

→ Treated case: Homogeneous potential with degree $2 < r < 6$.

[Ben1] M. Ben Said: Kramers–Fokker–Planck operators with homogeneous potentials. Math Meth Appl Sci. 2021;1–14. https://doi.org/10.1002/mma.7822

2. Is the equivalence

$$K_V \text{ has a compact resolvent} \Leftrightarrow \Delta_V^{(0)} \text{ has a compact resolvent} \quad (7.1)$$

true for a general potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$?

3. Study of the behaviour of the semi-group $(e^{-tK_V})_{t\geq 0}$ in cases not yet treated.

→ Some treated cases:

[Cat] P. Cattiaux, Long time behavior of Markov processes, ESAIM Proc. Vol 44, pp. 110-129, 2014.

[DoFoGu] R. Douc, G. Fort, A. Guillin, Subgeometric rates of convergence of f-ergodic strong Markov processes, Stochastic Process. Appl., 19(2009), 897-923.

[Wan] X. P. Wang, Large-time asymptotics of solutions to the Kramers-Fokker-Planck equation with a short-range potential. Comm. Math. Phys. 336 (2015), no. 3, 1435-1471.

[NoWa] R. Novak, X.P. Wang, On the Kramers-Fokker-Planck equation with decreasing potentials in dimension one, J. of Spectral Theory, 10(1) (2020), pp 1-32.

[WaZh] X. P. Wang AND L. ZHU, Global-in-time $L^p - L^q$ estimates for solutions of the Kramers-Fokker-Planck equation (2021) arxiv.org/abs/2107.03246.

Thank you for your attention !