Exponential stabilization of a flexible structure with dynamic delayed boundary conditions via one boundary control only

Sabeur Mansouri

IPEIM

Joint work with Boumediène Chentouf
Journal of The Franklin Institute

Plan of the talk

1. Introduction
2. Well-posedness
3. exponential stability

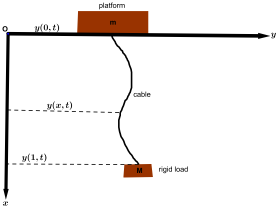
Plan of the talk

1. Introduction
2. Well-posedness
3. exponential stability

Plan of the talk

1. Introduction
2. Well-posedness
3. exponential stability

Description of the problem



Assumption

- *The cable is completely flexible and non-stretching,*
- *Transversal and angular displacements are small,*
- *Friction is neglected,*
- *The angle of the cable with respect to the vertical-axis is small everywhere.*

Notations

- $\vec{T}$: the tension of the cable,
- $\theta(x, t)$: the angle between $\vec{T}$ and the x-axis,
- Δx : the length of a portion of the cable.

Assumption

- *The cable is completely flexible and non-stretching,*
- *Transversal and angular displacements are small,*
- *Friction is neglected,*
- *The angle of the cable with respect to the vertical-axis is small everywhere.*

Notations

- $\vec{T}$: the tension of the cable,
- $\theta(x,t)$: the angle between $\vec{T}$ and the x-axis,
- Δx : the length of a portion of the cable.

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (ay_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \; t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$my_{tt}(0,t) = a(0)y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$My_{tt}(1,t) = -a(1)y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x(x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

Newton's law

$$\Downarrow$$

$$\Delta x y_{tt}(x,t) = T(x+\Delta x)\theta(x+\Delta x,t) - T(x)\theta(x,t).$$

$$\theta(x,t) \simeq y_x(x,t), \quad T(x) = (M+1-x)g,$$

EDP

$$y_{tt}(x,t) - (a y_x)_x (x,t) + \mathcal{F}(x,t) = 0, 0 < x < 1, \ t > 0,$$

where $a(x) = T(x) = (M+1-x)g$

EDO

$$m y_{tt}(0,t) = a(0) y_x(0,t) + \mathcal{G}_1(1,t), t > 0,$$

$$M y_{tt}(1,t) = -a(1) y_x(1,t) + \mathcal{G}_2(1,t), t > 0.$$

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) + \mathcal{F}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ my_{tt}(0,t) - (ay_x)(0,t) = \mathcal{G}_1(0,t), & t > 0, \\ My_{tt}(1,t) + (ay_x)(1,t) = \mathcal{G}_2(1,t), & t > 0, \end{cases} \tag{1}$$

où

$a \in H^1(0,1)$, and there exists a positive constant a_0 such that

$$a(x) \geq a_0 > 0, \quad \text{for all} \quad x \in [0,1]. \tag{2}$$

$$(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2)$$

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) + \mathcal{F}(x,t) = 0, & 0 < x < 1, \ t > 0, \\ my_{tt}(0,t) - (ay_x)(0,t) = \mathcal{G}_1(0,t), & t > 0, \\ My_{tt}(1,t) + (ay_x)(1,t) = \mathcal{G}_2(1,t), & t > 0, \end{cases} \tag{1}$$

où

$a \in H^1(0,1)$, and there exists a positive constant a_0 such that

$$a(x) \geq a_0 > 0, \quad \text{for all} \quad x \in [0,1]. \tag{2}$$

$(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2)$

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) + \mathcal{F}(x,t) = 0, & 0 < x < 1, \ t > 0, \\ my_{tt}(0,t) - (ay_x)(0,t) = \mathcal{G}_1(0,t), & t > 0, \\ My_{tt}(1,t) + (ay_x)(1,t) = \mathcal{G}_2(1,t), & t > 0, \end{cases} \tag{1}$$

où

$a \in H^1(0,1)$, and there exists a positive constant a_0 such that

$$a(x) \geq a_0 > 0, \quad \text{for all} \quad x \in [0,1]. \tag{2}$$

$$\left(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2\right)$$

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  $\alpha < \beta$. Polynomial convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = b(x) y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0, \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  $\alpha < \beta$. Polynomial convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = b(x) y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0, \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  $\alpha < \beta$. Polynomial convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0. \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = b(x) y_t(x,t), \\ \mathcal{G}_1(0,t) = -\beta y_t(0,t) + \alpha y_t(0,t-\tau), \\ \mathcal{G}_2(1,t) = 0, \end{cases}$$

  Exponential stability result.

- $$\begin{cases} \mathcal{F}(x,t) = (y_{xt})_x(x,t), \\ \mathcal{G}_1(0,t) = \alpha y_t(0,t-\tau) + \beta(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = -\beta(ay_{xt})(1,t) - \gamma y_t(1,t), \end{cases}$$

  Exponential convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\sigma y(0,t) + \nu y_t(0,t-\tau) - \zeta y_t(0,t) + \omega(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = 0, \end{cases} \tag{3}$$

  $$\begin{cases} \zeta = \delta + \sigma\omega, \\ m > \delta\omega, \end{cases} \tag{4}$$

- $$\begin{cases} \mathcal{F}(x,t) = (y_{xt})_x(x,t), \\ \mathcal{G}_1(0,t) = \alpha y_t(0,t-\tau) + \beta(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = -\beta(ay_{xt})(1,t) - \gamma y_t(1,t), \end{cases}$$

  Exponential convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\sigma y(0,t) + \nu y_t(0,t-\tau) - \zeta y_t(0,t) + \omega(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = 0, \end{cases} \tag{3}$$

$$\begin{cases} \zeta = \delta + \sigma\omega, \\ m > \delta\omega, \end{cases} \tag{4}$$

- $$\begin{cases} \mathcal{F}(x,t) = (y_{xt})_x(x,t), \\ \mathcal{G}_1(0,t) = \alpha y_t(0,t-\tau) + \beta(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = -\beta(ay_{xt})(1,t) - \gamma y_t(1,t), \end{cases}$$

  Exponential convergence result.

- $$\begin{cases} \mathcal{F}(x,t) = 0, \\ \mathcal{G}_1(0,t) = -\sigma y(0,t) + \nu y_t(0,t-\tau) - \zeta y_t(0,t) + \omega(ay_{xt})(0,t), \\ \mathcal{G}_2(1,t) = 0, \end{cases} \tag{3}$$

$$\begin{cases} \zeta = \delta + \sigma\omega, \\ m > \delta\omega, \end{cases} \tag{4}$$

$$\xi(t) = my_t(0,t) + \omega\left[\sigma y(0,t) - (ay_x)(0,t)\right].$$

$$E(t) = \frac{1}{2}\left[\int_0^1 \left[y_t^2(x,t) + a(x)y_x^2(x,t) + K\tau y_t^2(0,t-x\tau)\right]\,dx\right] + \frac{1}{2}\left[\sigma y^2(0,t) + \frac{1}{m-\delta\omega}\xi^2(t) + My_t^2(1,t)\right].$$

In order to make the energy $E(t)$ decreasing, we shall assume that

$$\nu < \min\left\{2, \frac{\delta}{1+\frac{\omega}{2(m-\delta\omega)}}\right\}, \tag{5}$$

and then choose K such that

$$\nu\left(1+\frac{\omega}{m-\delta\omega}\right) \leq K \leq 2\delta - \nu, \tag{6}$$

$$\xi(t) = my_t(0,t) + \omega\left[\sigma y(0,t) - (ay_x)(0,t)\right].$$

$$E(t) = \frac{1}{2}\left[\int_0^1 \left[y_t^2(x,t) + a(x)y_x^2(x,t) + K\tau y_t^2(0,t-x\tau)\right]\,dx\right] + \frac{1}{2}\left[\sigma y^2(0,t) + \frac{1}{m-\delta\omega}\xi^2(t) + My_t^2(1,t)\right].$$

In order to make the energy $E(t)$ decreasing, we shall assume that

$$\nu < \min\left\{2, \frac{\delta}{1+\frac{\omega}{2(m-\delta\omega)}}\right\}, \tag{5}$$

and then choose K such that

$$\nu\left(1+\frac{\omega}{m-\delta\omega}\right) \le K \le 2\delta - \nu, \tag{6}$$

$$\xi(t) = my_t(0,t) + \omega\left[\sigma y(0,t) - (ay_x)(0,t)\right].$$

$$E(t) = \frac{1}{2}\left[\int_0^1 \left[y_t^2(x,t) + a(x)y_x^2(x,t) + K\tau y_t^2(0,t-x\tau)\right]\,dx\right]$$
$$+\frac{1}{2}\left[\sigma y^2(0,t) + \frac{1}{m-\delta\omega}\xi^2(t) + My_t^2(1,t)\right].$$

In order to make the energy $E(t)$ decreasing, we shall assume that

$$\nu < \min\left\{2, \frac{\delta}{1+\frac{\omega}{2(m-\delta\omega)}}\right\}, \tag{5}$$

and then choose K such that

$$\nu\left(1+\frac{\omega}{m-\delta\omega}\right) \le K \le 2\delta - \nu, \tag{6}$$

$$\xi(t) = my_t(0,t) + \omega\left[\sigma y(0,t) - (ay_x)(0,t)\right].$$

$$E(t) = \frac{1}{2}\left[\int_0^1 \left[y_t^2(x,t) + a(x)y_x^2(x,t) + K\tau y_t^2(0,t-x\tau)\right]\,dx\right] + \frac{1}{2}\left[\sigma y^2(0,t) + \frac{1}{m-\delta\omega}\xi^2(t) + My_t^2(1,t)\right].$$

In order to make the energy $E(t)$ decreasing, we shall assume that

$$\nu < \min\left\{2, \frac{\delta}{1+\frac{\omega}{2(m-\delta\omega)}}\right\}, \tag{5}$$

and then choose K such that

$$\nu\left(1+\frac{\omega}{m-\delta\omega}\right) \leq K \leq 2\delta - \nu, \tag{6}$$

$$u(x,t) = y_t(0, t - x\tau),$$

which implies that the system becomes

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) = 0, & x \in I,\ t > 0, \\ \tau u_t(x,t) + u_x(x,t) = 0, & x \in I,\ t > 0, \\ my_{tt}(0,t) - (ay_x)(0,t) = -\sigma y(0,t) + \nu u(1,t) & \\ -(\delta + \sigma\omega)u(0,t) + \omega y_{xt}(0,t), & t > 0, \\ My_{tt}(1,t) + (ay_x)(1,t) = 0, & t > 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in I, \\ y_t(0, t - \tau) = f(t - \tau), & t \in (0, \tau). \end{cases} \tag{7}$$

$$u(x,t) = y_t(0, t - x\tau),$$

which implies that the system becomes

$$\begin{cases} y_{tt}(x,t) - (ay_x)_x(x,t) = 0, & x \in I, \ t > 0, \\ \tau u_t(x,t) + u_x(x,t) = 0, & x \in I, \ t > 0, \\ my_{tt}(0,t) - (ay_x)(0,t) = -\sigma y(0,t) + \nu u(1,t) & \\ -(\delta + \sigma\omega)u(0,t) + \omega y_{xt}(0,t), & t > 0, \\ My_{tt}(1,t) + (ay_x)(1,t) = 0, & t > 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), & x \in I, \\ y_t(0, t-\tau) = f(t-\tau), & t \in (0,\tau). \end{cases} \tag{7}$$

$$\mathcal{H} = H^1(I) \times L^2(I) \times L^2(I) \times \mathbb{R}^2,$$

$$\langle (y,z,u,\xi,\eta),(\tilde{y},\tilde{z},\tilde{u},\tilde{\xi},\tilde{\eta})\rangle_{\mathcal{H}} = \int_0^1 (a y_x \tilde{y}_x + z\tilde{z})\,dx + K\tau \int_0^1 u\tilde{u}\,dx$$
$$+\sigma y(0)\tilde{y}(0) + \frac{1}{m-\delta\omega}\xi\tilde{\xi} + M\eta\tilde{\eta},$$

Let $\Phi = (y,z,u,\xi,\eta)$, where $z(.,t) = y_t(.,t)$ and $\eta(t) = y_t(1,t)$.

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t), \\ \Phi(0) = \Phi_0, \end{cases}$$

$$\mathcal{H} = H^1(I) \times L^2(I) \times L^2(I) \times \mathbb{R}^2,$$

$$\langle (y,z,u,\xi,\eta),(\tilde{y},\tilde{z},\tilde{u},\tilde{\xi},\tilde{\eta})\rangle_{\mathcal{H}} = \int_0^1 (a y_x \tilde{y}_x + z\tilde{z})\, dx + K\tau \int_0^1 u\tilde{u}\, dx$$
$$+\sigma y(0)\tilde{y}(0) + \frac{1}{m-\delta\omega}\xi\tilde{\xi} + M\eta\tilde{\eta},$$

Let $\Phi = (y,z,u,\xi,\eta)$, where $z(.,t) = y_t(.,t)$ and $\eta(t) = y_t(1,t)$.

$$\begin{cases} \Phi_t(t) = \mathcal{A}\Phi(t), \\ \Phi(0) = \Phi_0, \end{cases}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (y, z, u, \xi, \eta) \in \mathcal{H}; y \in H^2(I),\ z, u \in H^1(I), \\ \\ \xi = mu(0) + \omega[\sigma y(0) - (ay_x)(0)], \\ u(0) = z(0),\ \eta = z(1) \end{array} \right\},$$

$$\mathcal{A}(y, z, u, \xi, \eta) =$$

$$\left(z, (ay_x)_x, -\frac{u_x}{\tau}, -\frac{1}{\omega}\xi + \frac{m - \delta\omega}{\omega} u(0) + \nu u(1), -\frac{(ay_x)(1)}{M} \right).$$

- $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$:

$$\begin{aligned}\rho : &= -\frac{1}{\omega}\xi + \frac{m-\delta\omega}{\omega}u(0) + \nu u(1)\\ &= -\sigma y(0) + \nu u(1) - \delta u(0) + (a y_x)(0).\end{aligned}$$

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} \leq \left(\frac{\nu}{2} + \frac{K}{2} - \delta\right) u^2(0) + \left(\frac{\nu}{2} - \frac{K}{2} + \frac{\nu\omega}{2(m-\delta\omega)}\right) u^2(1) + \left(\frac{\nu}{2} - 1\right) \frac{\omega}{m-\delta\omega}\rho^2.$$

- Lax-Milgram $\Rightarrow \mathrm{Im}(\lambda I - \mathcal{A}) = \mathcal{H}$.
- $\mathcal{A}$ generates on $\mathcal{H}$ a C_0-semigroup of contractions $e^{t\mathcal{A}}$.

Theorem

1. If $\Phi_0 \in \mathcal{D}(\mathcal{A})$,

$$\begin{gathered} y \in C^2\left(0, \infty; L^2(I)\right) \cap C^1\left(0, \infty; H^1(I)\right) \cap C\left(0, \infty; H^2(I)\right); \\ u \in C^1\left(0, \infty; L^2(I)\right) \cap C\left(0, \infty; H^1(I)\right); \\ \xi,\ \eta \in C^1\left(0, \infty; \mathbb{R}\right). \end{gathered}$$

2. If $\Phi_0 \in \mathcal{H}$,

$$\begin{gathered} y \in C^1\left(0, \infty; L^2(I)\right) \cap C\left(0, \infty; H^1(I)\right); \\ u \in C\left(0, \infty; L^2(I)\right); \\ \xi,\ \eta \in C\left(0, \infty; \mathbb{R}\right). \end{gathered}$$

3. The spectrum of $\mathcal{A}$ consists of isolated eigenvalues of finite algebraic multiplicity only.

Theorem

Suppose that (2), (4), (5) and (6) are fulfilled. The system (7) is exponential stable in $\mathcal{H}$, i.e., there exist $C > 0$ and $\varpi > 0$ such that for all $t > 0$ we have the following the decay estimate

$$\left\| e^{t\mathcal{A}}\Phi \right\|_{\mathcal{H}} \leq Ce^{-\varpi t} \left\| \Phi \right\|_{\mathcal{H}}, \quad \forall \Phi \in \mathcal{H}.$$

Theorem

Suppose that (2), (4), (5) and (6) are fulfilled. The system (7) is exponential stable in $\mathcal{H}$, i.e., there exist $C > 0$ and $\varpi > 0$ such that for all $t > 0$ we have the following the decay estimate

$$\left\| e^{t\mathcal{A}}\Phi \right\|_{\mathcal{H}} \leq Ce^{-\varpi t} \left\| \Phi \right\|_{\mathcal{H}}, \quad \forall \Phi \in \mathcal{H}.$$

Lemma 1

If λ is a real number, then $i\lambda$ is not an eigenvalue of $\mathcal{A}$.

$$\Downarrow$$

The system (7) is strongly stable, i.e.,

$$\lim_{t\to\infty} E(t) = 0.$$

Lemma 2

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{A}(\mathcal{H})} < \infty,$$

Proof: By contradiction
Banach-Steinhaus Theorem $\Rightarrow \exists\ \lambda_n \to +\infty$ and
$\Phi_n = (y_n, z_n, u_n, \xi_n, \eta_n) \in D(\mathcal{A})$ with

$$\|\Phi_n\|_{\mathcal{H}} = 1$$

such that

$$(i\lambda_n I - \mathcal{A})\Phi_n = (f_n, g_n, h_n, p_n, q_n) \to 0, \quad \text{as } n \to \infty \quad \text{in} \quad \mathcal{H}.$$

$$\begin{cases} i\lambda_n y_n - z_n & := \quad f_n \to 0, \quad \text{in} \quad H^1(I), \\ i\lambda_n z_n - (a(x)(y_n)_x)_x & := \quad g_n \to 0, \quad \text{in} \quad L^2(I), \\ i\lambda_n u_n + \dfrac{(u_n)_x}{\tau} & := \quad h_n \to 0, \quad \text{in} \quad L^2(I), \\ i\lambda_n \xi_n + \dfrac{1}{\omega}\xi_n - \dfrac{m - \delta\omega}{\omega} u_n(0) - \nu u_n(1) & := \quad p_n \to 0, \quad \text{in} \quad \mathbb{C}, \\ i\lambda_n \eta_n + \dfrac{(a(y_n)_x)(1)}{M} & := \quad q_n \to 0, \quad \text{in} \quad \mathbb{C}. \end{cases}$$

Proof: By contradiction
Banach-Steinhaus Theorem $\Rightarrow \exists\ \lambda_n \to +\infty$ and
$\Phi_n = (y_n, z_n, u_n, \xi_n, \eta_n) \in D(\mathcal{A})$ with

$$\|\Phi_n\|_{\mathcal{H}} = 1$$

such that

$$(i\lambda_n I - \mathcal{A})\Phi_n = (f_n, g_n, h_n, p_n, q_n) \to 0, \quad \text{as } n \to \infty \quad \text{in } \ \mathcal{H}.$$

$$\begin{cases} i\lambda_n y_n - z_n & := \quad f_n \to 0, \quad \text{in} \quad H^1(I), \\ i\lambda_n z_n - (a(x)(y_n)_x)_x & := \quad g_n \to 0, \quad \text{in} \quad L^2(I), \\ i\lambda_n u_n + \dfrac{(u_n)_x}{\tau} & := \quad h_n \to 0, \quad \text{in} \quad L^2(I), \\ i\lambda_n \xi_n + \dfrac{1}{\omega}\xi_n - \dfrac{m-\delta\omega}{\omega} u_n(0) - \nu u_n(1) & := \quad p_n \to 0, \quad \text{in} \quad \mathbb{C}, \\ i\lambda_n \eta_n + \dfrac{(a(y_n)_x)(1)}{M} & := \quad q_n \to 0, \quad \text{in} \quad \mathbb{C}. \end{cases}$$

$$-\lambda_n^2 y_n - (a(y_n)_x)_x = i\lambda_n f_n + g_n,$$

$$\theta(\overline{y}_n)_x,$$

where θ is the function in $C^1([0,1],\mathbb{R})$

$$\Downarrow$$

$$\int_0^1 (a\theta_x - a_x\theta)(x)|(y_n)_x|^2\,dx + \int_0^1 \lambda_n^2 \theta_x(x)|y_n(x)|^2\,dx$$

$$-a(1)\theta(1)|(y_n)_x(1)|^2 = o(1).$$

$$-\lambda_n^2 y_n - (a(y_n)_x)_x = i\lambda_n f_n + g_n,$$

$$\theta(\overline{y}_n)_x,$$

where θ is the function in $C^1([0,1],\mathbb{R})$

$$\Downarrow$$

$$\int_0^1 (a\theta_x - a_x\theta)(x)|(y_n)_x|^2\,dx + \int_0^1 \lambda_n^2\theta_x(x)|y_n(x)|^2\,dx$$

$$-a(1)\theta(1)|(y_n)_x(1)|^2 = o(1).$$

$$-\lambda_n^2 y_n - (a(y_n)_x)_x = i\lambda_n f_n + g_n,$$

$$\theta(\overline{y}_n)_x,$$

where θ is the function in $C^1([0,1],\mathbb{R})$

$$\Downarrow$$

$$\int_0^1 (a\theta_x - a_x\theta)(x)|(y_n)_x|^2\,dx + \int_0^1 \lambda_n^2\theta_x(x)|y_n(x)|^2\,dx$$

$$-a(1)\theta(1)|(y_n)_x(1)|^2 = o(1).$$

Now let us choose

$$\theta(x) = \exp\left(\int_x^1 \left|\frac{a_x}{a}\right|(s)\,ds\right)\left(-M - \int_x^1 \exp\left(-\int_s^1 \left|\frac{a_x}{a}\right|(r)\,dr\right)\,ds\right),$$

$$\begin{cases} \theta(1) = -M, \\ \theta_x(x) \geq 1, \ \forall x \in [0,1], \\ (a\theta_x - |a_x|\theta)(x) \geq a(x), \ \forall x \in [0,1], \\ \theta(x) < 0, \ \forall x \in [0,1] \end{cases}$$

$$\Downarrow$$

$$y_n \to 0 \quad \text{in} \quad H^1(I).$$

Now let us choose

$$\theta(x) = \exp\left(\int_x^1 \left|\frac{a_x}{a}\right|(s)\,ds\right)\left(-M - \int_x^1 \exp\left(-\int_s^1 \left|\frac{a_x}{a}\right|(r)\,dr\right)\,ds\right),$$

$$\begin{cases} \theta(1) = -M, \\ \theta_x(x) \geq 1,\ \forall x \in [0,1], \\ (a\theta_x - |a_x|\theta)(x) \geq a(x),\ \forall x \in [0,1], \\ \theta(x) < 0,\ \forall x \in [0,1] \end{cases}$$

$$\Downarrow$$

$$y_n \to 0 \quad \text{in} \quad H^1(I).$$

Now let us choose

$$\theta(x) = \exp\left(\int_x^1 \left|\frac{a_x}{a}\right|(s)\,ds\right)\left(-M - \int_x^1 \exp\left(-\int_s^1 \left|\frac{a_x}{a}\right|(r)\,dr\right)\,ds\right),$$

$$\begin{cases} \theta(1) = -M, \\ \theta_x(x) \geq 1,\ \forall x \in [0,1], \\ (a\theta_x - |a_x|\theta)(x) \geq a(x),\ \forall x \in [0,1], \\ \theta(x) < 0,\ \forall x \in [0,1] \end{cases}$$

$$\Downarrow$$

$$y_n \to 0 \quad \text{in} \quad H^1(I).$$

For $\Phi \in \mathcal{D}(\mathcal{A}^2)$, we consider

$$\mathcal{E}(t) = E(t) + \varepsilon \mathcal{V}(t), \text{ with } \mathcal{V}(t) = V_0(t) + V_1(t) + V_2(t),$$

$$\begin{cases} V_0(t) = 2\displaystyle\int_0^1 \theta(x) y_t y_x \, dx, \\ V_1(t) = K\tau \displaystyle\int_0^1 e^{-\vartheta\tau x} y_t^2(0, t - x\tau)\, dx, \ \vartheta \text{ is a constant}, \\ V_2(t) = C_2 y(0,t) \left(\displaystyle\int_0^1 y_t \, dx + M y_t(1,t) \right), \ C_2 \text{ to be chosen.} \end{cases}$$

$$A_1 E(t) \leq \mathcal{E}(t) \leq A_2 E(t),$$

$$A_1 = 1 - \varepsilon C_* \quad \text{and} \quad A_2 = 1 + \varepsilon C_*,$$

$$C_* = \left(2 \left\| \frac{\theta}{\sqrt{a}} \right\|_{L^\infty(I)} + 2 + \frac{\sqrt{M}+1}{\sqrt{\sigma}} C_2 \right).$$

$$\frac{d}{dt}\mathcal{E}(t) = ?$$

$$E'(t) \leq -B_1 y_t^2(0,t) - B_2 y_t^2(0,t-\tau) - B_3 \xi'^2(t),$$

with

$$B_1 = \delta - \frac{K+\nu}{2}, \quad B_2 = \frac{1}{2}\left(K - \nu - \frac{\nu\omega}{m - \delta\omega} \right)$$

$$\text{and } B_3 = \left(1 - \frac{\nu}{2}\right) \frac{\omega}{m - \delta\omega}.$$

$$A_1 E(t) \leq \mathcal{E}(t) \leq A_2 E(t),$$

$$A_1 = 1 - \varepsilon C_* \quad \text{and} \quad A_2 = 1 + \varepsilon C_*,$$

$$C_* = \left(2 \left\| \frac{\theta}{\sqrt{a}} \right\|_{L^\infty(I)} + 2 + \frac{\sqrt{M}+1}{\sqrt{\sigma}} C_2 \right).$$

$$\frac{d}{dt}\mathcal{E}(t) = ?$$

$$E'(t) \leq -B_1 y_t^2(0,t) - B_2 y_t^2(0,t-\tau) - B_3 \xi'^2(t),$$

with

$$B_1 = \delta - \frac{K+\nu}{2}, \quad B_2 = \frac{1}{2}\left(K - \nu - \frac{\nu\omega}{m-\delta\omega}\right)$$

$$\text{and } B_3 = \left(1 - \frac{\nu}{2}\right)\frac{\omega}{m-\delta\omega}.$$

$$A_1 E(t) \leq \mathcal{E}(t) \leq A_2 E(t),$$

$$A_1 = 1 - \varepsilon C_* \quad \text{and} \quad A_2 = 1 + \varepsilon C_*,$$

$$C_* = \left(2 \left\| \frac{\theta}{\sqrt{a}} \right\|_{L^\infty(I)} + 2 + \frac{\sqrt{M}+1}{\sqrt{\sigma}} C_2 \right).$$

$$\frac{d}{dt}\mathcal{E}(t) = ?$$

$$E'(t) \leq -B_1 y_t^2(0,t) - B_2 y_t^2(0,t-\tau) - B_3 \xi'^2(t),$$

with

$$B_1 = \delta - \frac{K+\nu}{2}, \quad B_2 = \frac{1}{2}\left(K - \nu - \frac{\nu\omega}{m-\delta\omega}\right)$$

$$\text{and } B_3 = \left(1 - \frac{\nu}{2}\right)\frac{\omega}{m-\delta\omega}.$$

$$A_1 E(t) \leq \mathcal{E}(t) \leq A_2 E(t),$$

$$A_1 = 1 - \varepsilon C_* \quad \text{and} \quad A_2 = 1 + \varepsilon C_*,$$

$$C_* = \left(2 \left\| \frac{\theta}{\sqrt{a}} \right\|_{L^\infty(I)} + 2 + \frac{\sqrt{M}+1}{\sqrt{\sigma}} C_2 \right).$$

$$\frac{d}{dt}\mathcal{E}(t) = ?$$

$$E'(t) \leq -B_1 y_t^2(0,t) - B_2 y_t^2(0,t-\tau) - B_3 \xi'^2(t),$$

with

$$B_1 = \delta - \frac{K+\nu}{2}, \quad B_2 = \frac{1}{2}\left(K - \nu - \frac{\nu\omega}{m - \delta\omega}\right)$$

$$\text{and } B_3 = \left(1 - \frac{\nu}{2}\right)\frac{\omega}{m - \delta\omega}.$$

$$A_1 E(t) \leq \mathcal{E}(t) \leq A_2 E(t),$$

$$A_1 = 1 - \varepsilon C_* \quad \text{and} \quad A_2 = 1 + \varepsilon C_*,$$

$$C_* = \left(2 \left\| \frac{\theta}{\sqrt{a}} \right\|_{L^\infty(I)} + 2 + \frac{\sqrt{M} + 1}{\sqrt{\sigma}} C_2 \right).$$

$$\frac{d}{dt}\mathcal{E}(t) = ?$$

$$E'(t) \leq -B_1 y_t^2(0,t) - B_2 y_t^2(0,t-\tau) - B_3 \xi'^2(t),$$

with

$$B_1 = \delta - \frac{K+\nu}{2}, \quad B_2 = \frac{1}{2}\left(K - \nu - \frac{\nu\omega}{m - \delta\omega}\right)$$

$$\text{and } B_3 = \left(1 - \frac{\nu}{2}\right)\frac{\omega}{m - \delta\omega}.$$

$$\begin{aligned}\mathcal{V}'(t) &\leq -\frac{1}{2}\int_0^1 y_t^2\,dx - \int_0^1 a y_x^2\,dx - \frac{M}{2}y_t^2(1,t) + \sigma^2\frac{\theta(0)}{a(0)}y^2(0,t)\\ &- K\vartheta e^{-\vartheta\tau}\tau\int_0^1 y_t^2(0,t-x\tau)\,dx - \frac{1}{2(m-\delta\omega)}\xi^2(t)\\ &+ K_1 y_t^2(0,t) + K_2 y_t^2(0,t-\tau) + K_3\xi'^2(t),\end{aligned} \tag{8}$$

where

$$K_1 = -\theta(0) - 3\frac{\theta(0)}{a(0)}\delta^2 + C_2^2\frac{M+1}{2} + K + \frac{m-\delta\omega}{2} + 1,$$

and

$$K_2 = -3\frac{\theta(0)}{a(0)}\nu^2 - Ke^{-\vartheta\tau} + \omega^2\nu^2\left(\frac{1}{m-\delta\omega} + \frac{1}{2}\right),$$

$$K_3 = -3\frac{\theta(0)}{a(0)} + \omega^2\left(\frac{1}{m-\delta\omega} + \frac{1}{2}\right).$$

$$\begin{aligned}\mathcal{V}'(t) &\leq -\frac{1}{2}\int_0^1 y_t^2\,dx - \int_0^1 a y_x^2\,dx - \frac{M}{2}y_t^2(1,t) + \sigma^2\frac{\theta(0)}{a(0)}y^2(0,t)\\ &- K\vartheta e^{-\vartheta\tau}\tau\int_0^1 y_t^2(0,t-x\tau)\,dx - \frac{1}{2(m-\delta\omega)}\xi^2(t)\\ &+ K_1 y_t^2(0,t) + K_2 y_t^2(0,t-\tau) + K_3\xi'^2(t),\end{aligned} \tag{8}$$

where

$$K_1 = -\theta(0) - 3\frac{\theta(0)}{a(0)}\delta^2 + C_2^2\frac{M+1}{2} + K + \frac{m-\delta\omega}{2} + 1,$$

and

$$K_2 = -3\frac{\theta(0)}{a(0)}\nu^2 - Ke^{-\vartheta\tau} + \omega^2\nu^2\left(\frac{1}{m-\delta\omega} + \frac{1}{2}\right),$$

$$K_3 = -3\frac{\theta(0)}{a(0)} + \omega^2\left(\frac{1}{m-\delta\omega} + \frac{1}{2}\right).$$

$$\varepsilon \leq \min\left\{\frac{B_1}{K_1}, \frac{B_2}{K_2}, \frac{B_3}{K_3}, \frac{1}{C_*}\right\},$$

$$\mathcal{E}'(t) \leq -c\varepsilon E(t) \leq -c\varepsilon A_2^{-1}\mathcal{E}(t).$$

$$c = \min\left\{1, 2\vartheta e^{-\vartheta\tau}, -2\sigma\frac{\theta(0)}{a(0)}\right\},$$

Theorem 2

Under the assumptions (2), (4), (5) and (6), the energy of the system (7) exponentially decays to zero with a decay rate

$$\varpi = A_2^{-1}\varepsilon c > 0.$$

$$\varepsilon \leq \min\left\{\frac{B_1}{K_1}, \frac{B_2}{K_2}, \frac{B_3}{K_3}, \frac{1}{C_*}\right\},$$

$$\mathcal{E}'(t) \leq -c\varepsilon E(t) \leq -c\varepsilon A_2^{-1}\mathcal{E}(t).$$

$$c = \min\left\{1, 2\vartheta e^{-\vartheta\tau}, -2\sigma\frac{\theta(0)}{a(0)}\right\},$$

Theorem 2

Under the assumptions (2), (4), (5) and (6), the energy of the system (7) exponentially decays to zero with a decay rate

$$\varpi = A_2^{-1}\varepsilon c > 0.$$

$$\varepsilon \leq \min\left\{\frac{B_1}{K_1}, \frac{B_2}{K_2}, \frac{B_3}{K_3}, \frac{1}{C_*}\right\},$$

$$\mathcal{E}'(t) \leq -c\varepsilon E(t) \leq -c\varepsilon A_2^{-1}\mathcal{E}(t).$$

$$c = \min\left\{1, 2\vartheta e^{-\vartheta\tau}, -2\sigma\frac{\theta(0)}{a(0)}\right\},$$

Theorem 2

Under the assumptions (2), (4), (5) and (6), the energy of the system (7) exponentially decays to zero with a decay rate

$$\varpi = A_2^{-1}\varepsilon c > 0.$$

$$\varepsilon \leq \min\left\{\frac{B_1}{K_1}, \frac{B_2}{K_2}, \frac{B_3}{K_3}, \frac{1}{C_*}\right\},$$

$$\mathcal{E}'(t) \leq -c\varepsilon E(t) \leq -c\varepsilon A_2^{-1}\mathcal{E}(t).$$

$$c = \min\left\{1, 2\vartheta e^{-\vartheta\tau}, -2\sigma\frac{\theta(0)}{a(0)}\right\},$$

Theorem 2

Under the assumptions (2), (4), (5) and (6), the energy of the system (7) exponentially decays to zero with a decay rate

$$\varpi = A_2^{-1}\varepsilon c > 0.$$

- F. Al-Musallam, K. Ammari and B. Chentouf, Asymptotic behavior of a 2D overhead crane with input delays in the boundary control, *Zeitschrift fur Angewandte Mathematik und Mechanik*, **98** (2018), 1103–1122.
- K. Ammari and B. Chentouf, Further results on the long-time behavior of a 2D overhead crane with a boundary delay: Exponential convergence, *Applied Mathematics and Computation*, **365** (2020).
- B. Chentouf and S. Mansouri, Exponential decay rate for the energy of a flexible structure with dynamic delayed boundary conditions and a local interior damping, *Applied Mathematics Letters*, **103** (2020).

- B. Chentouf and S. Mansouri, On the exponential stabilization of a flexible structure with dynamic delayed boundary conditions via one boundary control only, *Journal of The Franklin Institute*, 358 (2021), 934–962.
- A. Mifdal, Stabilisation uniforme d'un système hybride, *C. R. Acad. Sci. Paris. Série I.*, **324** (1997) 37–42.

MANY THANKS FOR YOUR KIND ATTENTION