# Control of coupled PDE-ODE and drilling optimization using regression models

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Control Theory Inverse Problems - CTIP 23 - Monastir, 08-10 May 2023

# OUTLINE

# Objectives

- **Modeling**:
  1. The first objective is to propose a solutions to avoid the torsional and axial vibrations.
  2. Basically, in our designs we use the backstepping techniques and the Lyapunov theory to establish the stability analysis.
- **Optimization of rate of penetration**
  We presents a comparison between three optimization methods, which is done by constructing mathematical models based on Eckel's equation, Galle and Woods equation and the regression model.

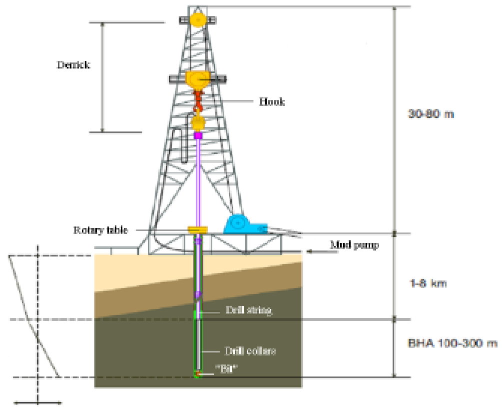
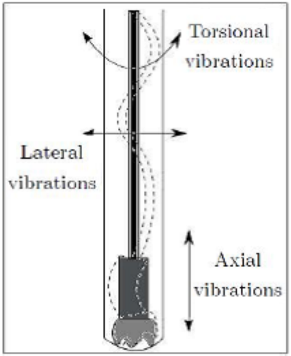



Figure: Drilling system

# Oilwell drilling system




Types of vibrations:

- torsional (stick-slip oscillations),
- axial (bit bouncing phenomenon),
- lateral (whirl motion due to the out-of-balance of the drillstring).

# Torsional model

The dynamic of the torsional variable $\theta(t,\xi)$ along the drillpipe is described by:

$$GJ\frac{\partial^2\theta}{\partial\xi^2}(t,\xi) - I\frac{\partial^2\theta}{\partial t^2}(t,\xi) - d\frac{\partial\theta}{\partial t}(t,\xi) = 0, \xi \in (0,L), t \in (0,+\infty). \quad (1)$$

with the boundary conditions

$$GJ\frac{\partial\theta}{\partial\xi}(t,0) = c_a\left(\frac{\partial\theta}{\partial t}(t,0) - \Omega(t)\right) \quad (2)$$

$$GJ\frac{\partial\theta}{\partial\xi}(t,L) - I_b\frac{\partial^2\theta}{\partial t^2}(t,L) = -R\left(\frac{\partial\theta}{\partial t}(t,L)\right) \quad (3)$$

where $I$ is the inertia, $G$ the shear modulus, $I_b$ inertia of the drillpipe, $J$ the geometrical moment of inertia, $d$ the drillstring damping, and $\Omega$ the control input (angular velocity due to the rotary table).

# Axial model

The axial dynamics of the drill string is described by an ordinary differential equation (ODE) as follows:

$$m_0\ddot{z}(t) + c_0[\dot{z}(t) + u_1(t)] + k_0 z(t) = -\mu_1 R(\frac{\partial\theta}{\partial t}(t, L)) \tag{4}$$

where:

- $z$ is defined as $z = Z - \Upsilon_0 t$, $m_0$, $c_0$ and $k_0$, denote the mass, damping and spring constant
- $Z$ represents the drill bit axial position.
- The dynamical system is controlled through the penetration rate $u_1(t)$, which is an axial speed imposed at the surface, and $\Upsilon_0$ is a constant value.

The coefficient $\mu_1$ can be modeled as $\mu_1 = \frac{2}{R_b m_{bit} c_{bit}}$ where $R_b$ is the bit radius, $\mu_{bit}$ is the friction coefficient at the bit-rock contact, and $c_{bit}$ is the bit coefficient.

# Coupled torsional-axial model

The coupling between both systems is due to the torque on bit $R$ approximating the physical phenomena at the bottom given by :

$$R(\frac{\partial\theta}{\partial t}(t,L)) = c_b\frac{\partial\theta}{\partial t}(t,L) + R_{nl}(\frac{\partial\theta}{\partial t}(t,L)) \tag{5}$$

The term $c_b\frac{\partial\theta}{\partial t}(t,L)$ represents the viscous damping torque which approximates the influence of the mud drilling and $R_{nl}(\frac{\partial\theta}{\partial t}(t,L))$ denotes the dry friction torque which models the bit-rock contact.

Hence, we obtain the next coupled torsional-axial vibrations:

$$GJ\frac{\partial^2\theta}{\partial\varsigma^2}(t,\varsigma) - I\frac{\partial^2\theta}{\partial t^2}(t,\varsigma) - d\frac{\partial\theta}{\partial t}(t,\varsigma) = 0 \tag{6}$$

$$GJ\frac{\partial\theta}{\partial\varsigma}(t,0) = c_a(\frac{\partial\theta}{\partial t}(t,0) - \Omega(t)) \tag{7}$$

$$GJ\frac{\partial\theta}{\partial\varsigma}(t,L) + I_b\frac{\partial^2\theta}{\partial t^2}(t,L) = -R(\frac{\partial\theta}{\partial t}(t,L)) \tag{8}$$

$$m_0\ddot{z}(t) + c_0[\dot{z}(t) + u_1(t)] + k_0 z(t) = -\mu_1 R(\frac{\partial\theta}{\partial t}(t,L)) \tag{9}$$

In order to improve clarity, we use the next variable change:

$$w(t,x) = \theta(L\sqrt{\frac{I}{GJ}}t, L(1-x)),\ x \in (0,1), \tag{10}$$

Then, we get

$$\partial_{tt}w(t,x) = \partial_{xx}w(t,x) - \beta\partial_t w(t,x), x \in (0,1) \tag{11}$$
$$\partial_x w(t,1) = u_0(t) \tag{12}$$
$$\partial_{tt}w(t,0) = a\partial_x w(t,0) + aF(\partial_t w(t,0)) \tag{13}$$
$$\ddot{z}(t) = -\frac{c_0}{m_0}[\dot{z}(t) + u_1(t)] - \frac{k_0}{m_0}z(t) + \mu_1\frac{GJ}{m_0 L}F(\partial_t w(t,0)) \tag{14}$$

where
$u_0(t) = \frac{c_a L}{GJ}\left(\Omega(t)(t) - \frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t w(t,1)\right)$, $\beta = dL\sqrt{\frac{1}{IGJ}}$,
$F\left(\partial_t w(t,0)\right) = -\frac{L}{GJ}R\left(\frac{1}{L}\sqrt{\frac{GJ}{I}}\partial_t w(t,0)\right)$, $a = \frac{LI}{I_b}$.
The controller $u_0(t)$ and $u_1(t)$ correspond to the angular velocity and the penetration rate imposed at the top end.

# Well posedness problem

Let $T > 0$, the solution of the Cauchy problem is written in this form

$$\partial_{tt}w(t,x) = \partial_{xx}w(t,x) - \beta\partial_t w(t,x) \quad (15)$$

$$\partial_x w(t,1) = u_0(t) \quad (16)$$

$$\partial_{tt}w(t,0) = a\partial_x w(t,0) + aF(\partial_t w(t,0)) \quad (17)$$

$$\ddot{z}(t) = -\frac{c_0}{m_0}\dot{z}(t) - \frac{c_0}{m_0}u_1(t) - \frac{k_0}{m_0}z(t) + \mu_1\frac{GJ}{m_0 L}F(\partial_t w(t,0)) \quad (18)$$

$$w(0,x) = w^0(x), \quad w_t(0,x) = w^1(x), z(0) = z^0, \dot{z}(0) = z^1 \quad (19)$$

where $x \in (0,1)$, $t \in (0,T)$, $w^0 \in E := \{w \in H^1(0,1), w^0(0) = 0\}$, $w^1 \in L^2(0,1)$. $z^0$ and $z^1$ are the imposed values to the solution at $t = 0$.

Let us propose the vector space $E$ which is equipped with the scalar product

$$< w^1(t,x), w^2(t,x) >_E = \int_0^1 w_x^1(t,x) w_x^2(t,x) dx \tag{20}$$

It is obvious that $E$ is a Hilbert space.
Let us consider $Z = (w(t,x), w_t(t,x), w_t(t,0), z(t), \dot{z})^T$. Equations (15)-(19) can be written as

$$\begin{aligned} \dot{Z}(t) &= AZ(t) + H(Z(t)) + f(t) && (21) \\ Z(0) &= Z_0 && (22) \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_{xx} & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-k_0}{m_0} & \frac{-c_0}{m_0} \end{pmatrix}, H(Z(t)) = \begin{pmatrix} 0 \\ 0 \\ aF(w_t(t,0)) \\ 0 \\ \mu_1 \frac{GJ}{m_0 L} F(\partial_t w(t,0)) \end{pmatrix},$$

and $f(t) = \begin{pmatrix} 0 \\ \delta(x-1)u_0(t) \\ 0 \\ 0 \\ u_1(t) \end{pmatrix}$

for which, $g = -a < \delta_0'(x), . >$ such that $\delta$ denotes the Dirac function, $\langle \delta_1'(x), w(t,x) \rangle = -w_x(t,1)$ and $\langle \delta_0'(x), w(t,x) \rangle = -w_x(t,0)$.

Our purpose in this section is to prove the existence and uniqueness of the system (21)-(22).

First, let us consider the linear problem of the system (21)-(22) (i.e $H(Z) = 0$ and $f(t) = 0$), we have the next theorem.
Let consider the next space $X = \{Z : w \in E,\ w_t \in L^2([0,1]),\ w_t(t,0) \in \mathbf{R},\ z \in \mathbf{R},\ \dot{z} \in \mathbf{R}\}$

**Theorem**

*The operator $A$ generates a $C_0$ semigroup $e^{At}$, $t \geq 0$ of contractions on $X$.*

In order to prove the existence of unique solution of the system (21)-(22), we need the next Lemma.

**Lemme**

*Let $z(t) \geq z(0)e^{-\alpha t}$ such that $\alpha = \frac{\frac{k_0}{m_0} - \sqrt{\Delta}}{2\frac{c_0}{m_0}}$ then*

$$z^2(t) - \frac{c_0}{m_0}(\dot{z}(t))^2 + \frac{k_0}{2m_0}\frac{d}{dt}z^2(t) \leq 0$$

In order to prove the existence and uniqueness of the nonlinear system (21)-(22) we introduce the next theorem.

## Theorem

*Let* $f \in L^1([0,T],X)$ *and* $Z_0 \in D(A)$, *then the problem* $\dot{Z}(t) = AZ(t) + H(Z) + f(t)$ *has a unique solution*

$$Z \in C^1([0,T],X) \bigcap C^0([0,T],D(A))$$

*given by:*

$$Z(t) = S(t)Z(0) + \int_0^t S(t-s)(H(Z(s)) + f(s))ds$$

To prove this Theorem, we need the next lemmas:

**Lemme**

*The nonlinear operator $H(Z)$ is dissipative and locally Lipschitz.*

**Lemme**

*For any function $f \in L^1([0,T],X)$, and any initial condition $Z_0 \in D(A)$, the problem (21)-(22) has at most one solution in $C^1([0,T],X) \cap C^0([0,T],D(A))$.*

# Stability of the controlled coupled system

In order to linearize the boundary condition (13), we introduce the next form:

$$\bar{w}(t,x) = \frac{\beta w_r}{2}x^2 - F(w_r)x + w_r t + w_0 \quad (23)$$

as a reference trajectory, such that $w_r = \partial_t \bar{w}(t,x)$.
Then the coupled PDE-ODE (11)-(13) becomes

$$\begin{aligned}
\partial_{tt} w(t,x) &= \partial_{xx} w(t,x) - \beta \partial_t w(t,x) && (24)\\
\partial_x w(t,1) &= u_0(t) && (25)\\
\partial_{tt} w(t,0) &= a\partial_x w(t,0) + ab\partial_t w(t,0) && (26)\\
\ddot{z}(t) &= -\frac{c_0}{m_0}\dot{z}(t) - \frac{c_0}{m_0}u_1(t) - \frac{k_0}{m_0}z(t) + \mu_1 \frac{GJ}{m_0 L}F(\partial_t w(t,0)) && (27)
\end{aligned}$$

where $u_0$ and $u_1$ are the control laws, $b = \frac{\partial F}{\partial s}(w_r)$ and $s(t) = \partial_t w(t,1)$.

Now, we establish the stability of the controlled coupled torsional-axial models.

**Theorem**

*Consider system (24)-(27), and the next both control laws*

$$\begin{aligned} u_0(t) &= \frac{1}{\partial_t w(t,1)}\left[(1-a)\partial_t w(t,0)\partial_x w(t,0) - ab\partial_t w(t,0)^2\right] \\ u_1(t) &= -\frac{k_0}{c_0} z(t) + \mu_1 \frac{GJ}{c_0 L} F(\partial_t w(t,0)) \end{aligned}$$

*Then the system (24)-(27) is stable at the equilibrium.*

# Neutral type delay coupled torsional-axial vibrations

We suppose that the damping term $\beta$ (given in (24)) is zero.
Using the d'Alembert's transformation to the distributed parameter model (24) to a difference equation model (neutral type delay system). Consequently, $w(t,x) = \eta(\sigma) + \nu(\gamma)$ is the general solution of the unidimensional wave equation (24) when $\beta = 0$, with $\sigma = t + x, \gamma = t - x$. We define

$$\dot{\varrho}(t) = \partial_t w(t,0) = \dot{\eta}(t) + \dot{\nu}(t), \tag{28}$$

as the angular velocity at the drill pipe bottom extremity. We introduce this into the boundary conditions (24)-(26), which gives

$$\dot{\eta}(t+1) - \dot{\nu}(t-1) = \tau_4(t) \tag{29}$$

$$\ddot{\eta}(t) + \ddot{\nu}(t) = a(\dot{\eta}(t) - \dot{\nu}(t)) + ab(\dot{\eta}(t) + \dot{\nu}(t)) \tag{30}$$

Then, we may define the next

$$\ddot{\varrho}(t) = a(b-1)\dot{\varrho}(t) + a(1+b)\dot{\varrho}(t-2) - \ddot{\varrho}(t-2) + 2a\tau_4(t-1)$$

Now, the coupled torsional-axial dynamic stability is analyzed through

$$\begin{aligned}\ddot{\varrho}(t) &= a(b-1)\dot{\varrho}(t) + a(1+b)\dot{\varrho}(t-2) - \ddot{\varrho}(t-2) + 2a\tau_4(t-1)\\ \ddot{z}(t) &= -\frac{m_0}{c_0}[\dot{z}(t) + u_1(t)] - \frac{k_0}{c_0}z(t) + \mu_1\frac{m_0 GJ}{c_0 L}F(\partial_t w(t,0))\end{aligned}$$

Let us defines $x_1 = \varrho$, $x_2 = \dot{\varrho}$. Then, we have:

$$\dot{x}_1(t) = x_2(t) \tag{31}$$

$$\dot{x}_2(t) = a(b-1)x_2(t) + a(1+b)x_2(t-2) - \dot{x}_2(t-2) + 2a\tau_4(t-1) \tag{32}$$

$$\ddot{z}(t) = -\frac{c_0}{m_0}\dot{z}(t) - \frac{c_0}{m_0}u_1(t) - \frac{k_0}{m_0}z(t) + \mu_1\frac{GJ}{m_0 L}F(\partial_t w(t,0)) \tag{33}$$

The main idea is to study the behavior position and displacement of the angular velocity.

**Theorem**

*Consider system (31)-(33) and the both control laws*

$$\tau_4(t-1) = -\frac{1}{2a}(x_1 + abx_2(t) + a(1+b)x_2(t-2) - \dot{x}_2(t-2)) \quad (34)$$

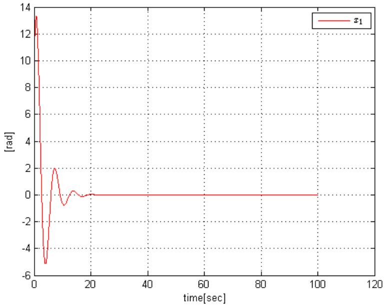
$$u_1(t) = -\frac{k_0}{m_0}z(t) + \mu_1 \frac{GJ}{c_0 L}F(\partial_t w(t,0)) \quad (35)$$

*Then the zero equilibrium is stable.*

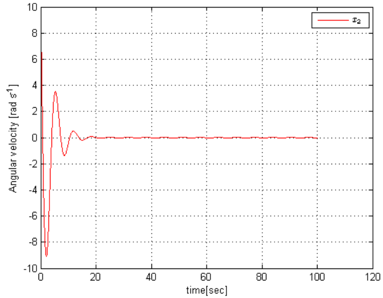
# Numerical simulations

The next physical parameters are used in simulation:

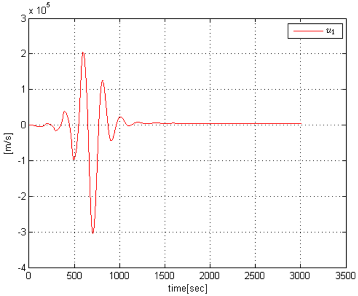
| Variable | Value | Description |
|---|---|---|
| $L$ | $2000\ [m]$ | drillstring length |
| $I$ | $0.095\ [kg.m]$ | inertia per unit length |
| $I_b$ | $311\ [kg.m^2]$ | inertia at the drillstring bottom |
| $J$ | $1.19.10^{-5}\ [m^4]$ | geometrical moment of inertia |
| $c_a$ | $2000\ [Nm.s.rad^{-1}]$ | sliding torque coefficient |
| $G$ | $79.3.10^9\ [N.m^{-2}]$ | shear modulus |
| $k_0$ | $1.55 \times 10^6\ [kg.s^{-2}]$ | spring constant |
| $m_0$ | $37278\ kg$ | mass |
| $c_0$ | $16100\ kgs^{-1}$ | damping |
| $\mu_1$ | $257\ [m^{-1}]$ | |



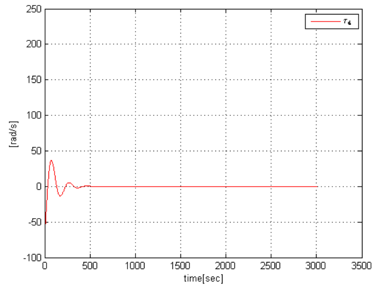

Angular position $x_1(t) \approx w(t,0)$




Angular velocity $x_2(t) = \partial_t w(t,0)$

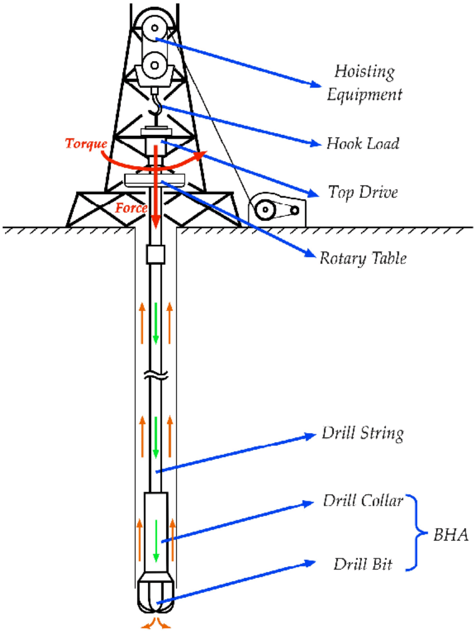



Control law $u_1(t)$




Control law $\tau_4(t)$

# Drilling Models




- Drilling parameters measured on the rig can be naively classified into control parameters,uncontrollable parameters, and response parameters.
- Control parameters are drilling operational parameters which can be controlled by the drilling engineer on the rig: WOB, RPM, and flow rate.
- Uncontrollable parameters are those which cannot be changed by engineers while drilling a well such as the strength of the rock, geological properties, maximum pump power.

The response (or objective) can then be optimized by changing the controllable parameters. This section shows how the objective functions can be modeled in terms of controllable and uncontrollable parameters using three different models :

- Eckel's model
- Galle and woods model
- Regression model

# Coefficient of determination

We should know that there are many criteria for the evaluation of conceptual models that we build such as :

- The multiple coefficient of determination,
- root mean squared error ,
- residual diagnostics and goodness of fit tests. . .

In our work, we use the multiple coefficient of determination.This variable is given by the next equation:

$$R^2 = 1 - \sum_{k=0}^{n} \frac{(y - \hat{y})^2}{(y - \bar{y})^2}$$

where :

- $y$:The actual output
- $\hat{y}$ :The output given by the model.
- $\bar{y}$ :The actual output mathematical average.

# Coefficient of determination

The score here is the $R^2$ score, which measures how well a model performs relative to a simple mean of the target values.

- $R^2 = 1$ indicates a perfect match,
- $R^2 = 0$ indicates the model does no better than simply taking the mean of the data, and negative values mean even worse models.

# Data base

| Depth(ft) | ROP(m/hr) | Predicted ROP(m/hr) | WOB(klbs) | RPM(tr/min) |
|---|---|---|---|---|
| 2170 | 3.352 | 3.303 | 13.06 | 96.4 |
| 2180 | 3.04799 | 2.661 | 8.64 | 104.4 |
| 2190 | 2.1336 | 2.869 | 9.86 | 103.4 |
| 2200 | 2.1336 | 3.626 | 14.62 | 102.8 |
| 2210 | 3.45 | 3.595 | 14.7 | 99.8 |
| 2220 | 3.048 | 3.036 | 11.06 | 100.2 |
| 2230 | 2.4384 | 2.57 | 8.36 | 100.6 |
| 2240 | 2.384 | 3.582 | 15.1 | 95 |
| 2250 | 2.1372 | 3.037 | 12.46 | 90.4 |
| 2260 | 2.04 | 2.282 | 7.74 | 92 |

Figure: Data base

# Eckel's model

The Eckel's model is described as following :

$$ROP = aWOB^b RPM^c$$

Where a, b and c are constants.

- ROP: forward speed (m / hr).
- WOB: the weight on the tool.
- RPM: the speed of rotation.
- a, b, c: the coefficients depending on the training

Using the least squared method then we are Searching for the minimum of the expression by deriving the least squares prescription (S)

$$S = log(a.WOB^b.RPM^c)$$

We calculate the coefficients of Eckel's equation using the least squares method.Then we obtain:

- $a = 0.032$
- $b = 1.56$
- $c = 0.283$

Consequently, Eckel's model is given by the following equation :

$$ROP = 0.0.32WOB^{1.56}RPM^{0.283}$$

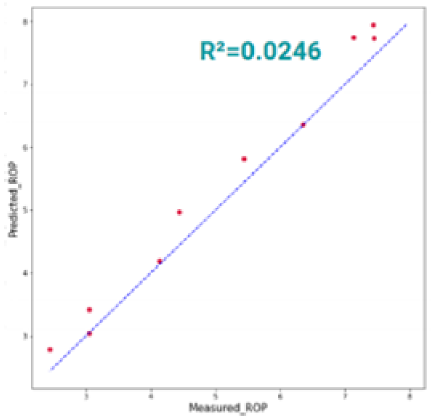



Figure: Predicted ROP vs Measured ROP

We observe in there is a clear improvement in penetration speed when the weight on the tool is optimum and optimum rotational speed are used.Using a created python based program predicted ROP is 13.45% higher than the real ROP in other words the Predicted ROP will take less time to reach the depth predicted as the actual ROP. Moreover Eckel's model gives a determination coefficient equal to $R^2 = 0.0246$

## Galle And Wood's model

- Galle and Woods (1963) were also among the first to examine the influence of the optimal constant bit weight and rotational speed for the lowest cost, as well as create mathematical relationships.
- To identify the optimal combinations of constant weight and rotation speed, graphs and processes for field applications were created.
- They assumed a relationship between the wear rate and the inverse ratio of bit weight to bit diameter as a function of time.
- They also presented an equation that relates tooth wear rate to rotational speed for milled tooth bits suited for soft formations alone.

Hence, the model is described as following:

$$ROP = C_f \frac{(WOB^k RPM^\alpha)}{a^p} \tag{36}$$

- The value of the weight exponent $k$ is related with the type of the formation: $0.6$ for very soft formation and $1.5$ for a hard one
- $k = 0.8$: a constant linked to the nature of the drilled formation.
- $\alpha = 0.4$: can vary from $0.4$ in hard terrains to $0.8$ in soft soils
- For $Cf = 0.37518$: we can determinate it using the data base and the equation (2)
- $C_f$ is the unknown variable and we extract the other variable from our data base .
- The adopted model was used to calculate the penetration rate at each point of data and the calculated ROP was plotted against the ROP measured , the coefficient of determination

$$R^2 = 0.2204$$

  which indicates that the Galle Woods model gives a better results compared to the Eckel's.

# Regression model

- The problem that arises for the engineer is the following. He has a set of measurements of variables of a process of any kind (physical, chemical, economic, financial, ...), and the result of this process.
- He assumes that there is a deterministic relation between these variables and this result, and he looks for a mathematical form of this relation.
- According to the point cloud, the forward speed can take the following form:

$$ROP = a + bWOB + cRPM + dWOB^2 + eRPM^2 \tag{37}$$

Where $a, b, c, d$ and $e$ are constants.

- We can calculate the coefficients of the regression equation using the least method squares and we re-inject the parameters applied in the equation (29) to calculate the forward speed from the model in order to test its accuracy.

$$ROP = 2863.6 + 93.3WOB - 62.57RPM - 1.702WOB^2 + 0.317RPM^2 (3)$$

- The adopted model was used to calculate the penetration rate at each point of data,
- The coefficient of determination

$$R^2 = 0.8316$$

which indicates that the ROP model gives good results

# Best model

- At the same interval, we must apply the three models designated above(Eckel's model, Galle and wood model and the regression model) so as to determine the most suitable model.
- The multiple coefficient of determination $R^2$ can be classified as numerical indicators to define an optimal model.
- We put the result of interval [2170ft, 2260ft] in a table to facilitate the choice of the model.

Table: Comparaison of three models.

| ECKEL Model | Galle and wood Model | Regression Model |
|---|---|---|
| $R^2 = 0.0246$ | $R^2 = 0.2204$ | $R^2 = 0.8316$ |

# Conclusion

- A new neutral coupled time-delay torsional-axial system has been introduced
- We offer a method for optimizing drilling mechanical parameters.
- The recommended optimization models are regression models that allow for the modification of drilling parameters depending on the type of the formations to be penetrated in order to maximize penetration rate.
- To the performance of automatic drilling applications it is necessary some variables such that the downhole pressure need to be estimated.