Variational approximation of interface energies for topology optimization and optimal partitioning

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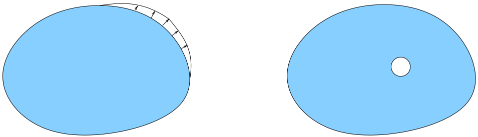
Review of classical tools in shape / topology optimization

- Shape derivative: consider a displacement $x \in \Omega \mapsto x + \theta(x)$

$$J((Id+\theta)(\Omega)) - J(\Omega) = \int_{\partial\Omega} g_S \theta \cdot n + o(\|\theta\|_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)})$$

- Topological derivative: consider a small topology perturbation, typically $\Omega_\varepsilon = \Omega \setminus \overline{B(z,\varepsilon)}$ and an expansion like

$$J(\Omega_\varepsilon) - J(\Omega) = \varepsilon^d g_T(z) + o(\varepsilon^d)$$



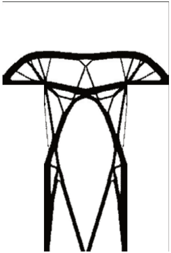
Shape vs topology perturbation

- Homogenization: incorporate intermediate (anisotropic) materials, obtained by "mixing" the strong and weak ($\approx$ void) phases $\rightsquigarrow$ existence of optimal designs.
Simplification: interpolation (e.g. SIMP)

Perimeter penalization in topology optimization

What for?

- To control the complexity of domains.
- To enforce the existence of optimal shapes because $BV(D) \hookrightarrow L^1(D)$ is compact.
- To model surface tensions.



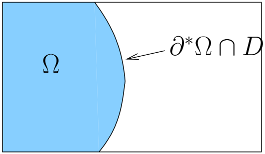
Difficulty

- The perimeter is differentiable w.r.t. smooth shape variations (shape derivative = mean curvature).
- For a topology perturbation of form $\Omega_\varepsilon = \Omega \setminus \overline{B(z,\varepsilon)}$, $\Omega \subset \mathbb{R}^d$, the perimeter varies like ε^{d-1}, while usual cost functions vary like ε^d (no topological derivative).

Perimeter in the sense of geometric measure theory

Let $D \subset \mathbb{R}^d$, open, bounded,

$\Omega \subset D$.



The relative perimeter of Ω in D is the Hausdorff measure

$$\boxed{\text{Per}_D(\Omega) = \mathcal{H}^{d-1}(\partial^*\Omega \cap D),}$$

where $\partial^*\Omega$ is the essential boundary of Ω (points of density different from 0 and 1 $\rightsquigarrow \partial^*\Omega \subset \partial\Omega$).

We also have:

$$\text{Per}_D(\Omega) = \int_D |D\chi_\Omega| = \sup\left\{\int_\Omega \text{div}\varphi, \varphi \in \mathcal{C}_c^1(D, \mathbb{R}^d), \|\varphi\|_\infty \leq 1\right\},$$

$\text{Per}_D(\Omega) < \infty \Leftrightarrow \chi_\Omega \in BV(D)$: set of finite perimeter.

Perimeter approximation by Γ-convergence

Γ-convergence (De Giorgi-Franzoni, 1975)

Definition

Let $F_n, F : X \to \mathbb{R}$, X metric space.

One says that $F_n \xrightarrow{\Gamma} F$ at $x \in X$ iif

1. $\forall x_n \to x,\ F(x) \leq \liminf F_n(x_n)$,
2. $\exists y_n \to x,\ F(x) \geq \limsup F_n(y_n)$.

Theorem

Suppose that

1. $F_n \xrightarrow{\Gamma} F$ *in* X,
2. $F_n(x_n) \leq \inf_X F_n + \varepsilon_n,\ \varepsilon_n \to 0$,
3. $x_n \to x$.

Then x *is a minimizer of* F *and* $\lim F_n(x_n) = F(x)$.

Remarks

- The convergence of (x_n) is usually obtained from an equicoercivity argument:

$$\sup\{F_n(x_n)\} < \infty \Rightarrow (x_n) \text{ is compact.}$$

  This property may be as difficult to prove as the Γ-convergence.
- If $F_n \xrightarrow{\Gamma} F$ and G is continuous then

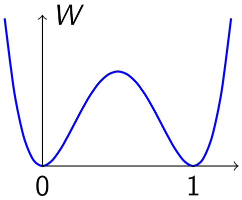
$$F_n + G \xrightarrow{\Gamma} F + G.$$

- The Γ-convergence does not imply the pointwise convergence $F_n(x) \to F(x)$.

A classical (local) perimeter approximation: the Van Der Waals-Cahn-Hiliard functional

For a potential $W : \mathbb{R} \to \mathbb{R}_+$ with wells 0 and 1 define

$$F_\varepsilon(u) = \int_D \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u).$$



u: density / phase field

Theorem (Modica-Mortola, 1977)

When $\varepsilon \to 0$ we have the Γ-convergence

$$\Gamma - \lim F_\varepsilon(u) = \begin{cases} c Per_D(\{u = 1\}) & \text{if } u \in BV(D, \{0,1\}) \\ +\infty & \text{otherwise} \end{cases}$$

in $L^1(D)$, with $c = \int_0^1 \sqrt{W(t)} dt$.

Advantages

- Approximation of the perimeter in the appropriate sense for optimization.
- Intermediate densities are penalized
  $\rightsquigarrow$ possible combination with relaxation / interpolation methods

Drawbacks

- The functional does not accept characteristic functions.
- The derivative w.r.t. u involves $-\Delta u$. Hence optimization by an explicit gradient method in L^2 may be very slow for fine grids (CFL condition).
  Using an H^1 scalar product raises difficulties for projecting onto $\{u \geq 0\}$.

These drawbacks stem from the term ∇u.

A non-local perimeter approximation

For all $u \in L^\infty(D,[0,1])$ consider $L_\varepsilon u := v_\varepsilon$ the smoothed version of u by

$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D, \\ \partial_n v_\varepsilon = 0 & \text{on } \partial D, \end{cases}$$

and define

$$\boxed{\tilde{F}_\varepsilon(u) := \frac{1}{\varepsilon}\int_D L_\varepsilon u(1-u) = \frac{1}{\varepsilon}\int_D (1-L_\varepsilon u)u.}$$

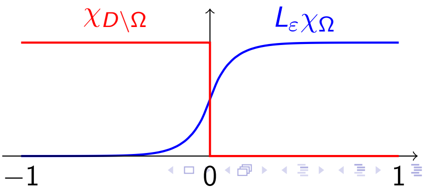
We have in particular

$$\tilde{F}_\varepsilon(\chi_\Omega) = \frac{1}{\varepsilon}\int_D (L_\varepsilon \chi_\Omega)\chi_{D\setminus\Omega}.$$

Example in 1d

$D = (-1,1)$, $\Omega = (0,1)$
One finds

$$\lim_{\varepsilon\to 0} \tilde{F}_\varepsilon(\chi_\Omega) = \frac{1}{2} = \frac{1}{2}\mathrm{Per}_D(\Omega).$$



Theorem (Γ-convergence and equicoercivity)

(*i*) *When* $\varepsilon \to 0$ *one has in* $L^1(D,[0,1])$

$$\Gamma - \lim \tilde{F}_\varepsilon(u) = \begin{cases} \frac{1}{2} Per_D(\{u=1\}) & \textit{if } u \in BV(D,\{0,1\}) \\ +\infty & \textit{otherwise.} \end{cases}$$

(*ii*) *If* $\sup_{\varepsilon>0} \tilde{F}_\varepsilon(u_\varepsilon) < \infty$ *then* (u_ε) *is compact in* $L^1(D,[0,1])$.

Remarks

- We have the variational formulation

$$\tilde{F}_\varepsilon(u) = \inf_{v \in H^1(D)} \left\{ \varepsilon \|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon}\left(\|v\|^2_{L^2(D)} + \int_D u(1-2v) \right) \right\}.$$

- In both expressions there is no ∇u.
- One also has the pointwise convergence $\tilde{F}_\varepsilon(\chi_\Omega) \to \frac{1}{2}\mathrm{Per}_D(\Omega)$.

Variant: heat kernel (Merriman-Bence-Osher, Miranda-Pallara-Paronnetto-Preunkert, Esedoglu-Otto), no variational form

Solution of topology optimization problems with perimeter penalization

Let $\tilde{J} : L^1(D, [0, 1]) \to \mathbb{R}$ be continuous and bounded from below,

$$I := \inf_{\Omega \subset D} \left\{ \tilde{J}(\chi_\Omega) + \frac{\alpha}{2} \mathrm{Per}_D(\Omega) \right\},$$

$$I_\varepsilon := \inf_{u \in L^\infty(D,[0,1])} \left\{ \tilde{J}(u) + \alpha \tilde{F}_\varepsilon(u) \right\}.$$

Γ-convergence and equicoercivity yield:

Theorem

Let u_ε be an approximate minimizer of I_ε, i.e.

$$\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \leq I_\varepsilon + \lambda_\varepsilon, \qquad \lambda_\varepsilon \to 0.$$

Then $\tilde{J}(u_\varepsilon) + \alpha \tilde{F}_\varepsilon(u_\varepsilon) \to I$.
Moreover, (u_ε) admits cluster points, and if u is a cluster point then $u = \chi_\Omega$ where Ω is a minimizer of I.

Bonus: convergence of derivatives

$$D\tilde{F}_\varepsilon(u)h = \frac{1}{\varepsilon}\int_D (1 - 2L_\varepsilon u)h =: \int_D g_{u,\varepsilon} h$$

Theorem

Let $\Omega \subset D$ and $x \in \partial\Omega \cap D$ such that $\partial\Omega$ is smooth around x. Then

$$g_{\chi_\Omega,\varepsilon}(x) \to \frac{1}{2}\kappa(x)$$

with $\kappa(x)$ the mean curvature of $\partial\Omega$ at x (shape derivative of the perimeter).

Examples

Conductivity maximization $\boxed{\gamma_\Omega = \gamma_0\chi_{D\setminus\Omega} + \gamma_1\chi_\Omega}$

$$J(\chi_\Omega) = \int_{\Gamma_N} gT + \ell|\Omega|, \quad \begin{cases} -\operatorname{div}(\gamma_\Omega \nabla T) = 0 & \text{in } D \\ \gamma_\Omega \nabla T.n = g & \text{on } \Gamma_N \end{cases}$$

We use the dual formulation of the thermal compliance

$$\int_{\Gamma_N} gT = \inf_{\substack{-\operatorname{div}\tau = 0 \\ \tau.n = g}} \int_D \gamma_\Omega^{-1}|\tau|^2.$$

Optimization: relaxation + alternating algorithm based on

$$I_\varepsilon = \inf_{u\in L^\infty(D,[0,1])} \inf_{v\in H^1(D)} \inf_{\substack{-\operatorname{div}\tau = 0 \\ \tau.n = g}} \Bigg\{ \int_D (\gamma_0(1-u) + \gamma_1 u)^{-1}|\tau|^2 + \ell \int_D u + \alpha\left[\varepsilon\|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon}\left(\|v\|^2_{L^2(D)} + \langle u, 1-2v\rangle\right)\right]\Bigg\}.$$

- Minimization w.r.t. τ amounts to solving the conductivity problem with $\gamma_u = \gamma_0(1-u) + \gamma_1 u$.
- Minimization w.r.t. u is given by

$$u = \begin{cases} 1 \text{ if } \ell + \dfrac{\alpha}{2\varepsilon}(1-2v) \le 0, \\ P_{[0,1]}\left(\sqrt{\dfrac{|\tau|^2}{(\gamma_1-\gamma_0)\left(\ell + \frac{\alpha}{2\varepsilon}(1-2v)\right)}} - \dfrac{\gamma_0}{\gamma_1-\gamma_0}\right) \text{ else.} \end{cases}$$

We consider a decreasing sequence (ε_k) from $\varepsilon_{\max} \approx \mathrm{diam}(D)$ to $\varepsilon_{\min} \approx h$.

Optimal heater for $\alpha = 0.1, 0.5, 2$, respectively ($\gamma_1 = 1$, $\gamma_0 = 10^{-3}$).

Compliance minimization in linear elasticity

Isotropic Hooke's tensor $\boxed{A_\Omega = \chi_{D\setminus\Omega} A_0 + \chi_\Omega A_1}$, $A_0 \approx 0$

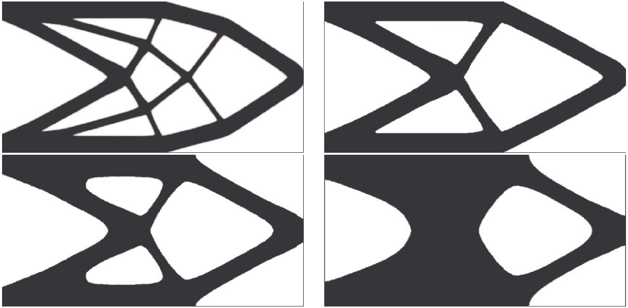
$$J(\chi_\Omega) = \int_{\Gamma_N} g \cdot y + \ell|\Omega|, \quad \begin{cases} -\operatorname{div}(A_\Omega \nabla^s y) = 0 & \text{in } D \\ A_\Omega \nabla^s y . n = g & \text{on } \Gamma_N \end{cases}$$

Relaxation $\Rightarrow$ homogenization (compliance in 2d $\rightsquigarrow$ rank 2 laminates)

$$\tilde{I}_\varepsilon = \inf_{u \in L^\infty(D,[0,1])} \inf_{v \in H^1(D)} \inf_{\substack{-\operatorname{div}\sigma = 0 \\ \sigma n = g}} \Bigg\{ \int_D A_1^{-1} \sigma : \sigma + \frac{1-u}{u} f^*(\sigma)$$
$$+ \ell \int_D u + \alpha \left[\varepsilon \|\nabla v\|^2_{L^2(D)} + \frac{1}{\varepsilon} \left(\|v\|^2_{L^2(D)} + \langle u, 1-2v \rangle \right) \right] \Bigg\}$$

Lamination formulas $\rightsquigarrow$ $f^*(\sigma)$ explicit

- Minimization w.r.t. σ $\Leftrightarrow$ find optimal material (standard homogenization, explicit)+ solve (anisotropic) elasticity system.
- Minimization w.r.t. u is again explicit.



Cantilever for $\alpha = 0.1, 2, 20, 50$, respectively.

Remark: boundary artefacts due to relative perimeter!

Case of unknown relaxation

We can use a level-set representation.

$$\boxed{\Omega_\psi = \{x \in D, \psi(x) > 0\}}$$

to solve the necessary optimality conditions:

$$\begin{cases} g_T \geq 0 \text{ in } \Omega_\psi & \text{(topology)} \\ g_S = 0 \text{ on } \partial\Omega_\psi \cap D & \text{(geometry)} \end{cases}$$

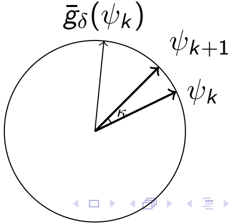
We can construct an approximation (close to SIMP!), then normalize:

$$g_\delta \underset{\delta \to 0}{\longrightarrow} \begin{cases} g_T \text{ in } \Omega_\psi \\ g_S \text{ on } \partial\Omega_\psi \cap D, \end{cases} \qquad \bar{g}_\delta = \frac{g_\delta}{\|g_\delta\|_{L^2(D)}}.$$

We perform damped fixed point iterations on the unit sphere of $L^2(D)$:

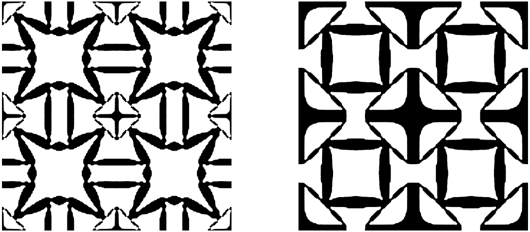
$$\psi_{k+1} = C_\kappa(\psi_k, \bar{g}_\delta(\psi_k))$$

where κ is found by line search (descent direction).



Example: optimal design of microstructures

Goal: optimize the Representative Volume Element to obtain desired homogenized properties (periodic model)



Poisson ratio minimization without (left) and with (right) perimeter penalization (periodic boundary condition, isotropy constraint).

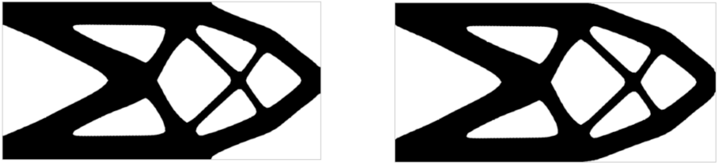
Variant: total perimeter

$$\mathrm{Per}^T(\Omega) := \mathcal{H}^{d-1}(\partial^*\Omega)$$
$$= \int_{\mathbb{R}^d} |D\chi_\Omega| = \sup\left\{ \int_D \chi_\Omega \operatorname{div}\varphi, \varphi \in \mathcal{C}^1(\bar{D}, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

It suffices to replace the Neumann boundary condition in L_ε by a Robin one:

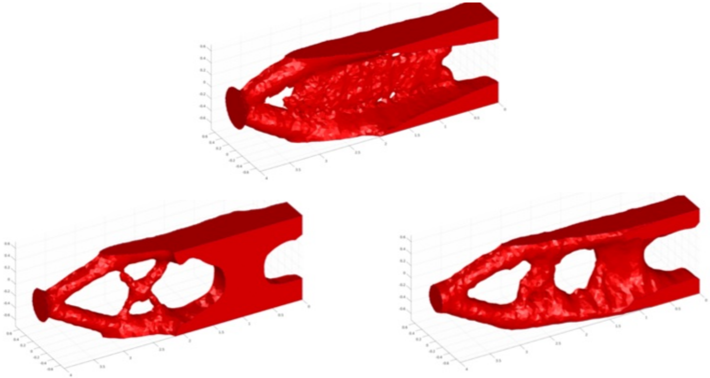
$$\begin{cases} -\varepsilon^2 \Delta v_\varepsilon + v_\varepsilon = u & \text{in } D, \\ \varepsilon \partial_n v_\varepsilon + v_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

Example: compliance minimization (level-set method)



Compliance minimization with relative (left) and total (right) perimeter penalization.

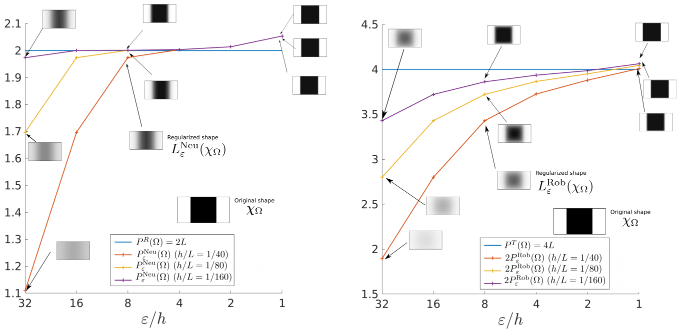
The same works in 3d



3d cantilever: without perimeter (top),
with relative perimeter (left), with total perimeter (right)

The 3d perimeter does not like plates!
Other geometric criteria may be of interest.

Numerical convergence



left: relative perimeter, right: total perimeter

Videos

Again the level set method, while ε is decreased...

• Perimeter minimization under volume constraint (projection / bisection)
Square with relative perimeter
Square with total perimeter

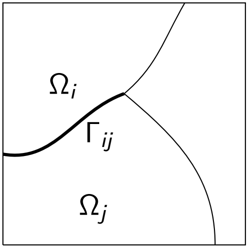
• Compliance minimization under volume constraint
Cantilever with relative perimeter
Cantilever with total perimeter

• Poisson ration minimization
Negative Poisson ratio with periodic perimeter

Extension: minimal partitions with interface energies

$$D = \Omega_1 \cup \cdots \cup \Omega_N$$

$$\Gamma_{ij} = \partial^* \Omega_i \cap \partial^* \Omega_j$$



Goal: minimize

$$\boxed{\sum_i \int_{\Omega_i} g_i + \sum_{i<j} \alpha_{ij} \mathcal{H}^{d-1}(\Gamma_{ij} \cap D)}$$

We have the pointwise approximation

$$\begin{aligned}
\mathcal{H}^{d-1}(\Gamma_{ij} \cap D) &= \frac{1}{2}\big[\mathcal{H}^{d-1}(\partial^* \Omega_i \cap D) + \mathcal{H}^{d-1}(\partial^* \Omega_j \cap D) \\
&\quad - \mathcal{H}^{d-1}(\partial^*(\Omega_i \cup \Omega_j) \cap D)\big] \\
&= \lim_{\varepsilon \to 0} \left[\tilde{F}_\varepsilon(\chi_{\Omega_i}) + \tilde{F}_\varepsilon(\chi_{\Omega_j}) - \tilde{F}_\varepsilon(\chi_{\Omega_i} + \chi_{\Omega_j})\right] \\
\cdots &= \lim_{\varepsilon \to 0} \frac{2}{\varepsilon} \int_D L_\varepsilon \chi_{\Omega_i} \chi_{\Omega_j}.
\end{aligned}$$

Questions: Γ-convergence? variational formulation?

Preliminary condition: lower semicontinuity

Theorem (Ambrosio-Braides, 1990)

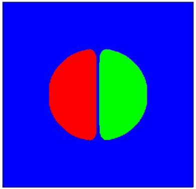
The triangle inequality

$$(T) \qquad \alpha_{ij} \leq \alpha_{ik} + \alpha_{kj} \qquad \forall i, j, k.$$

is necessary and sufficient for the functional

$$F : (\Omega_1, ..., \Omega_N) \mapsto \sum_{i<j} \alpha_{ij} \mathcal{H}^{d-1}(\Gamma_{ij} \cap D)$$

to be lower semicontinuous (w.r.t. convergence in measure).



$\alpha_{red/green} > \alpha_{red/blue} + \alpha_{green/blue}$

$\Rightarrow$ lack of lower semicontinuity

Γ-convergence

The work space is

$$S = \left\{ (u_1, \cdots, u_N) \in L^1(D, [0,1])^N : \sum_{i=1}^{N} u_i = 1 \right\}.$$

Γ-convergence can be proven under different sets of assumptions, in particular:

Theorem

If D is a Cartesian product of intervals, condition (T) implies the Γ-convergence of the functional

$$(u_1, \cdots, u_N) \in S \mapsto \frac{1}{\varepsilon} \sum_{i<j} \alpha_{ij} \int_D L_\varepsilon u_i \; u_j .$$

Convexity issues

Consider the symmetric matrix $Q = (\alpha_{ij})$.

Definition

We say that Q is conditionally negative semidefinite ($Q \preceq 0$) if

$$\sum_{ij} \alpha_{ij}\xi_i\xi_j \leq 0 \qquad \forall \xi \in \mathbb{R}^N \ : \ \sum_i \xi_i = 0.$$

Theorem

If $N \leq 4$ and (T) is fulfilled, then $Q \preceq 0$.
If $Q \preceq 0$ then $\mathcal{I}_\varepsilon$ is concave on

$$V := \left\{ u \in L^2(D, \mathbb{R}^N) \ : \ \sum_{i=1}^N u_i = 1 \right\}.$$

Consequence: Legendre duality

$$-\mathcal{I}_\varepsilon + \delta_V = (-\mathcal{I}_\varepsilon + \delta_V)^{**}$$

This leads to the variational formulation

$$\boxed{\mathcal{I}_\varepsilon(u) = \inf_{\sum_i v_i = 1} \frac{1}{\varepsilon} \sum_{i,j} \alpha_{ij} \left(\langle u_i, v_j \rangle - \frac{1}{2}\varepsilon^2 \langle \nabla v_i, \nabla v_j \rangle - \frac{1}{2} \langle v_i, v_j \rangle \right).}$$

For minimizing $\sum_i \int_D g_i u_i + \mathcal{I}_\varepsilon(u)$ we again suggest alternating minimizations:

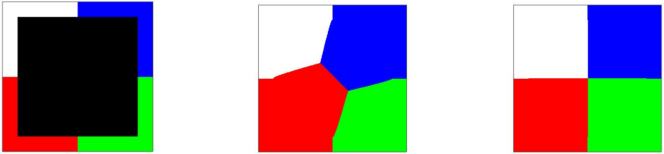
- $v_i = L_\varepsilon u_i$
- explicit (linear spatially separated) minimization w.r.t. u.

Remark: If $Q \succeq 0$ (general case by additive decomposition) we obtain by another duality scheme

$$\mathcal{I}_\varepsilon(u) = \frac{1}{\varepsilon} \inf_{\tau \in [H_0^{\mathrm{div}}(\Omega)]^N} \sum_{i,j=1}^{N} \alpha_{ij} \int_\Omega \tau_i \cdot \tau_j dx + \sum_{i,j=1}^{N} \alpha_{ij} \int_\Omega (u_i - \varepsilon \,\mathrm{div}\tau_i)(u_j - \varepsilon \,\mathrm{div}\tau_j) dx.$$

Example

Given a partition $(E_0, E_1, ..., E_N)$ set $g_i = 1 - \chi_{E_i}$: phase i is favoured in E_i, $i \geq 1$.



Partition with 4 phases: data E_i (left), obtained result for $\alpha_{ij} = 1\ \forall i, j$ (middle), obtained result for $\alpha_{ij} = 1$ if E_i and E_j are adjacent and $\alpha_{ij} = 2$ otherwise (right)

Volume constraints

We consider the constraints

$$\int_D u_i dx = m_i \ \forall i = 1, ..., N.$$

The minimization w.r.t. u is spatially coupled: it yields a linear programming subproblem of form

$$\min_{\substack{\sum_{i=1}^N u_i = 1 \\ u_i \geq 0, \ \int_\Omega u_i dx = m_i}} \qquad \Lambda(u) = \sum_{i=1}^N \int_\Omega \zeta_i u_i dx.$$

Since N is small compared with the number of pixels and the unconstrained problem is straightforward we consider the Lagrangian dual criterion

$$\mathcal{D}(\lambda) = \inf_{\substack{\sum_{i=1}^N u_i = 1 \\ u_i \geq 0}} \Lambda(u) + \sum_{i=1}^N \lambda_i \left(\int_\Omega u_i dx - m_i \right)$$

$$= \int_\Omega \min\{(\zeta_i + \lambda_i)_{i=1}^N\} - \sum_{i=1}^N \lambda_i m_i.$$

Theorem

The N-tuple $(\lambda_1, \dots, \lambda_N)$ *is a maximizer of* $\mathcal{D}$ *if and only if each* λ_i *is a maximizer of the partial function*

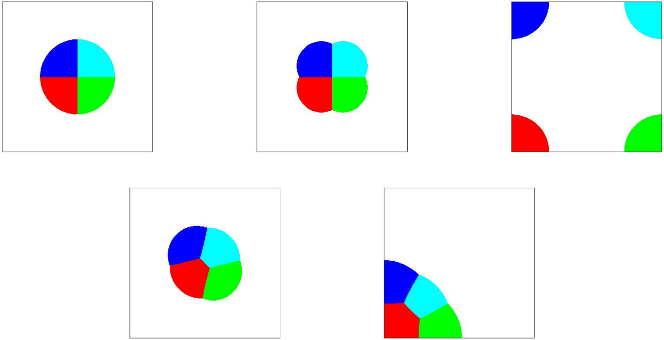
$$\tilde{\lambda}_i \mapsto \mathcal{D}(\lambda_1, \dots, \lambda_{i-1}, \tilde{\lambda}_i, \lambda_{i+1}, \dots, \lambda_N).$$

This is also equivalent to satisfying for each $i = 1, \dots, N$

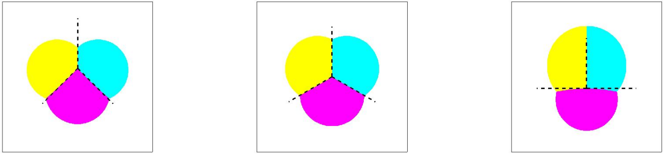
$$|\{\lambda_i < \min_{j \neq i}(\zeta_j + \lambda_j) - \zeta_i\}| \leq m_i \leq |\{\lambda_i \leq \min_{j \neq i}(\zeta_j + \lambda_j) - \zeta_i\}|.$$

Conclusion: alternating maximizations can be used at the cost of sorting pixels at each iteration.

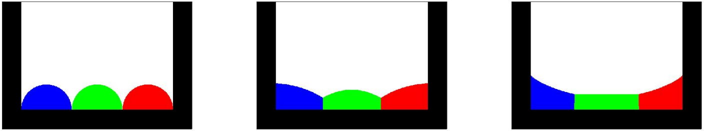
Examples



Partition with 5 phases and volume constraints:
initialization (top left), result for $\alpha_{ij} = 1 \ \forall i, j$ (top middle),
result for $\alpha_{int/int} = 1$ and $\alpha_{int/ext} = 0.5$ (top right),
result for $\alpha_{int/int} = 1$ and $\alpha_{int/ext} = 2$ (bottom left),
same case with $\varepsilon_{\max}$ large (bottom right).



Verification of Herring's theoretical angles
(generalization of the Fermat point).



Partition of 3 liquid phases + vapor + solid (fixed):
initialization (left), result for $\alpha_{ij} = 1 \ \forall i, j$ (center),
result for $\alpha_{LL} = 0.5$, $\alpha_{LS} = 1$, $\alpha_{LV} = \alpha_{SV} = 2$ (right).

Anisotropic perimeter

Let K be a closed, bounded, convex subset of $\mathbb{R}^d$. To simplify the presentation we assume $\operatorname{int} K \neq \emptyset$. Define

$$\mathrm{Per}_D^{\rho,K}(\Omega) = \int_{\partial^*\Omega\cap D} \rho(x)\sigma_K(\nu)\,d\mathcal{H}^{d-1},$$

$$\tilde{F}_\varepsilon(u) := \inf_{v\in H^1(D)} \varepsilon\rho^2\sigma_K^2(\nabla v) + \frac{1}{\varepsilon}\big(v^2 + u(1-2v)\big).$$

σ_K: support function of K; ν : inner normal

Theorem

When $\varepsilon \to 0$ one has in $L^1(D,[0,1])$

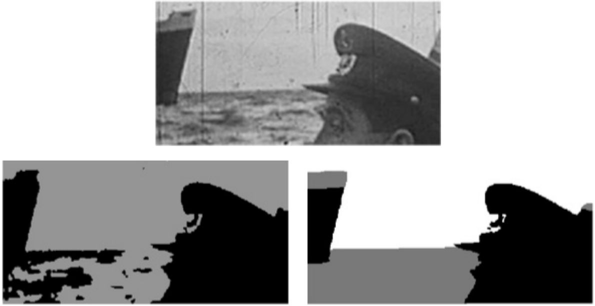
$$\Gamma - \lim \tilde{F}_\varepsilon(u) = \begin{cases} \frac{1}{2} Per_D^{\rho,K}(\{u=1\}) & \text{if } u \in BV(D,\{0,1\}) \\ +\infty & \text{otherwise.} \end{cases}$$

Application: unsupervised image classification

We take K as an ellipse $\rightsquigarrow$ linear PDE

Original image f $\rightarrow$ segmented image $w = \sum_i u_i c_i$

$$\text{by minimizing} \qquad \|w - f\|_{L^2}^2 + \frac{\alpha}{2} \sum_{i=1}^{N} \mathrm{Per}_D^{\rho,K}(\Omega_i).$$



Original image (top), 3 phase classification with isotropic perimeter (left), 3 phase classification with anisotropic perimeter (right)

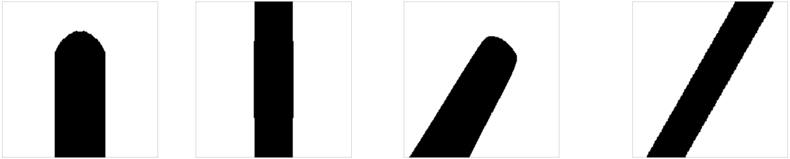
K is the ellipse of center 0 and axes 100 - 1.

Nonlinear case

Consider the segment

$$K = \{t\vec{k}, -\beta \le t \le \alpha\}.$$

We use proximal splitting.



$\alpha = 1,\ \beta = 0.1$ $\vec{k} = (0, 1)$ | $\alpha = 1,\ \beta = 1$ $\vec{k} = (0, 1)$ | $\alpha = 1,\ \beta = 0.1$ $\vec{k} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$ | $\alpha = 1,\ \beta = 1$ $\vec{k} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$

Sets minimizing the anisotropic perimeter given by a segment

Perspective:

Penalization of vertical downward normals (overhangs) for the design of 3D printed parts.