Global-in-time $L^p - L^q$ estimates for the Kramers-Fokker-Planck equation

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The Kramers-Fokker-Planck equation

The Kramers-Fokker-Planck equation is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field:

$$\frac{\partial W}{\partial t} = \left(-v \cdot \nabla_x + \nabla_v \cdot (\gamma v - \frac{F(x)}{m}) + \frac{\gamma kT}{m} \Delta_v \right) W,$$

where $F(x) = -m \nabla_x V(x)$ is the external force and $W = W(t; x, v)$ is the distribution function of particles for $x, v \in \mathbb{R}^n$ and $t > 0$.

This equation is also called the Kramers equation (H.A. Kramers (1940)) or the Fokker-Planck equation.

The Kramers-Fokker-Planck equation

After appropriate normalisation of physical constants and change of unknowns, the KFP equation can be written into the form

$$\partial_t u(t;x,v) + Pu(t;x,v) = 0,\ (x,v) \in \mathbb{R}^n \times \mathbb{R}^n, t > 0, \qquad (1)$$

with initial data

$$u(0;x,v) = u_0(x,v). \qquad (2)$$

P is the KFP operator defined by

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - \nabla_x V(x) \cdot \nabla_v.$$

Denote $P_0 = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x$.

The Kramers-Fokker-Planck equation

Assume

$$|V(x)| + \langle x\rangle |\nabla_x V(x)| \leq C\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^n, \tag{3}$$

for some $\rho > -1$. The potential is decreasing if $\rho > 0$ and slowly increasing if $-1 < \rho < 0$).

When $\rho > -1$ the potential term $\nabla_x V(x) \cdot \nabla_v$ is relatively compact w.r.t. P_0. Therefore one may study the KFP equation by *scattering methods*.

The Kramers-Fokker-Planck equation

Let $\mathfrak{m}$ be the function defined by

$$\mathfrak{m}(x,v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(\frac{v^2}{2}+V(x))}.$$

Then $\mathfrak{M} = \mathfrak{m}^2$ is the Maxwellian and $\mathfrak{m}$ verifies the stationary KFP equation

$$P\mathfrak{m} = 0 \quad \text{in } \mathbb{R}^{2n}_{x,v}.$$

When $\mathfrak{M}$ can be normalized in L^1, it is the (global) equilibrium. Otherwise, $\mathfrak{M}$ can be interpreted as a local equilibrium.

Some known results

The large-time behavior of solutions of the KFP equation is mostly studied for confining potentials :

$$V(x) \geq C\langle x\rangle^{1+\epsilon}, \quad |\nabla_x V(x)| \geq C\langle x\rangle^{\epsilon}$$

for $|x|$ large. In this case, 0 is a discrete eigenvalue of P. The typical result is return to the equilibrium with exponential rate: $\exists c > 0$ such that

$$u(t) = \langle \mathfrak{m}, u_0\rangle \mathfrak{m} + O(e^{-ct}), \quad t \to +\infty,$$

where $V(x)$ is assumed to be normalized by

$$\int_{\mathbb{R}^n} e^{-V(x)} dx = 1.$$

Some known results

For weakly confining potential $V(x) \sim \langle x \rangle^\sigma$, $|x| \to \infty$, $0 < \sigma < 1$, $(\rho = -\sigma)$, 0 is an eigenvalue embedded in the essential spectrum of P. T. Li and Z.F. Zhang (2018) proved the convergence to equilibrium with sub-exponential rate :

$$u(t) = \langle \mathfrak{m}, u_0 \rangle \mathfrak{m} + O(e^{-ct^{\frac{\sigma}{2-\sigma}}}), \quad t \to +\infty,$$

Some known results

For decreasing potentials, it is shown by W. (2015) for $n = 3$ and R. Novak-W. (2020) for $n = 1$ that

$$u(t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(\langle \mathfrak{m}, u_0 \rangle \mathfrak{m} + O(t^{-\epsilon}) \right), \quad t \to +\infty,$$

in weighted L^2-spaces with weight in x-variables.

Notation

In this work, we consider potentials $V(x)$ satisfying (3) with $\rho > 1$ and study $L^p - L^q$ estimates of $u(t)$ for $t > 0$, where

$$L^p = L^p(\mathbb{R}^{2n}_{x,v}; dxdv).$$

For $f \in L^p$ and T bounded linear operator from L^p to L^q, we denote :

$$\|f\|_p = \|f\|_{L^p}, \quad \|T\|_{p\to q} = \|T\|_{\mathcal{L}(L^p,L^q)}. \tag{4}$$

Notation

For a closed linear operator T in L^2 with $C_0^\infty(\mathbb{R}^{2n})$ as a core and for $p \in [1, \infty[$, we still denote by the same letter T its minimal closed extension in L^p (*i.e.*, the closure in L^p of the restriction of T to $C_0^\infty(\mathbb{R}^{2n})$).

Under fairly general condition, e^{-tP} is a strongly continuous positivity preserving contraction semigroup in L^p. Since for $1 \leq p < \infty$,

$$\overline{\left(e^{-tP}|_{C_0^\infty}\right)}|_{L^p} = e^{-t\,\overline{(P|_{C_0^\infty})}|_{L^p}},$$

our notation is consistent in some sense.

The main result

Theorem 1 (Zhu LU (Hohai Univ., Nanjing) -W.)

Let $n = 3$ *and condition (3) be satisfied with* $\rho > 1$. *For* $1 \leq p < q \leq \infty$, *there exists some constant* $C > 0$ *such that*

$$\|e^{-tP}\|_{p \to q} \leq \frac{C}{(\gamma(t))^{\frac{3}{2p}(1-\frac{p}{q})}}, \quad t \in]0, \infty[, \tag{5}$$

where $\gamma(t) = \sigma(t)\theta(t)$ *with*

$$\sigma(t) = t - 2\coth(t) + 2\mathrm{cosech}(t), \quad \theta(t) = 4\pi e^{-t}\sinh(t). \tag{6}$$

$\gamma(t) \sim ct^4$ as $t \to 0$ and $\gamma(t) \sim c't$ as $t \to \infty$.

The main result

$\theta(t)$ is related to the semigroup generated by the harmonic oscillator

$$H = \Re P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}, \quad v \in \mathbb{R}^3.$$

For $p = 1, q = \infty$,

$$(\sigma(t))^{-\frac{3}{2}} = O(t^{-\frac{3}{2}}), \quad t \to \infty; (\sigma(t))^{-\frac{3}{2}} = O(t^{-\frac{9}{2}}), \quad t \to 0_+.$$

This term can be compared with $e^{t\Delta_x}$ as map from $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ as $t \to \infty$ and with of $e^{-t|D_x|^{\frac{2}{3}}}$ as $t \to 0$. This result may be explained by the fact that at low energies, P behaves like a Witten Laplacian (B. Helffer-F. Nier, W.X. Li, $\cdots$), while globally P is sub-elliptic in x with the loss of $\frac{1}{3}$ derivatives.

A comment

For decreasing potentials, it is also natural to study the KFP equation in $\mathcal{L}^p$ spaces, where

$$\mathcal{L}^p = L^2(\mathbb{R}^n_v; L^p(\mathbb{R}^n_x)).$$

$(W(x, v, t) = \mathfrak{m}(x, v)u(x, v, t))$. One can show that for $\delta > 0$,

$$e^{-\delta P}: \quad L^p \to \mathcal{L}^p, \quad \mathcal{L}^p \to L^p$$

is bounded. Theorem 1 implies

$$\|e^{-tP}\|_{\mathcal{L}^p \to \mathcal{L}^q} \le \frac{C}{t^{\frac{3}{2p}(1-\frac{p}{q})}}, \quad t \ge 1. \tag{7}$$

Method of the proof

Method to prove Theorem 1 :

- Study first the free semigroup e^{-tP_0} in $L^p - L^q$ setting, where

$$P_0 = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x .$$

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- Consider P as perturbation of P_0 and use Duhamel's formula. The main task is to estimate e^{-tP} when t large.

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- Consider P as perturbation of P_0 and use Duhamel's formula. The main task is to estimate e^{-tP} when t large.
- The method still works for $n \geq 4$ (although not written).

The free KFP operator

Let P_0 be the free KFP operator:

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}, (x, v) \in \mathbb{R}^{2n}. \tag{8}$$

One has

$$P_0 u(x,v) = \mathcal{F}^{-1}_{x \to \xi} \widehat{P}_0(\xi) \widehat{u}(\xi, v), \quad \text{where} \tag{9}$$

$$\widehat{P}_0(\xi) = -\Delta_v + \frac{1}{4} \sum_{j=1}^{n} (v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2 \tag{10}$$

$$\widehat{u}(\xi, v) = (\mathcal{F}_{x \to \xi} u)(\xi, v) \triangleq \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x, v)\, dx. \tag{11}$$

The free KFP operator

Denote

$$D(\widehat{P}_0) = \{f \in L^2(\mathbb{R}^{2n}_{\xi,v}); \widehat{P}_0(\xi)f \in L^2(\mathbb{R}^{2n}_{\xi,v})\}. \quad (12)$$

Then $\widehat{P}_0 \triangleq \mathcal{F}_{x\to\xi} P_0 \mathcal{F}^{-1}_{x\to\xi}$ is a direct integral of the family of complex harmonic operators $\{\widehat{P}_0(\xi); \xi \in \mathbb{R}^n\}$. $\{\widehat{P}_0(\xi), \xi \in \mathbb{R}^n\}$ is a holomorphic family of type (A).

The distributional kernel of e^{-tP_0} can be deduced from Melher's formula by complex deformation.

The free KFP operator

Lemma 2

Let $n \geq 1$. The distributional kernel of e^{-tP_0} is given by

$$F(x,v,x',v';t) = \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} e^{-\frac{1}{4\sigma(t)}|x-x'-\omega(t)(v+v')|^2} K(v,v';t).$$

where

$$\begin{aligned} K(v,v';t) &= \frac{1}{(\theta(t))^{\frac{n}{2}}} e^{-\frac{\coth(t)}{4}(|v|^2+|v'|^2)+\frac{\operatorname{cosech}(t)}{2}v\cdot v'} \\ \omega(t) &= \coth(t) - \operatorname{cosech}(t). \end{aligned}$$

The free KFP operator

$K(v, v'; t)$ is the distributional kernel of e^{-tH}, $H = -\Delta_v + \frac{1}{4}v^2 - \frac{n}{2}$.

The fundamental solution $F(x, v, x', v'; t)$ for the free KFP equation has several nice properties. For example, one has for $f \in C_0^\infty(\mathbb{R}^{2n})$,

$$|\int (e^{-tP_0} f)(x, v)dx| \le (e^{-tH}g)(v), \quad v \in \mathbb{R}^n, \tag{13}$$

where $g(v) = \int |f(x', v)|dx'$.

The free KFP operator

From the formula of $F(x, v, x', v'; t)$, one obtains

Proposition 1

Let $n \geq 1$. For $t > 0$, e^{-tP_0} defined on $C_0^\infty(\mathbb{R}^{2n})$ extends to an operator bounded from L^1 to L^∞ and the following estimate is true for the free KFP operator:

$$\|e^{-tP_0}\|_{1\to\infty} \leq \frac{1}{(4\pi\gamma(t))^{\frac{n}{2}}} \tag{14}$$

for $t > 0$. Here

$$\gamma(t) = \sigma(t)\theta(t), \quad \theta(t) = 4\pi e^{-t}\sinh(t).$$

The free KFP operator

Proposition 2

One has

$$\|e^{-tP_0}\|_{p\to p} \leq 1 \tag{15}$$

for $1 \leq p \leq \infty$ *and*

$$\|e^{-tP_0}\|_{L^p\to L^q} \leq \frac{1}{(4\pi\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}}, \quad t > 0, \tag{16}$$

for $1 \leq p \leq q \leq \infty$. e^{-tP_0}, $t \geq 0$, *is a strongly continuous positivity preserving contraction semigroup in* L^p *for* $1 \leq p < \infty$.

The free KFP operator

Proof. By explicit calculation, one has

$$\|e^{-tP_0}f\|_1 \leq \|e^{-tH}f\|_1 \leq \|f\|_1$$
$$\|(e^{-tP_0} - 1)f\|_1 \leq \|(e^{-tH} - 1)f\|_1$$

for $f \in L^1$. This shows that $\|e^{-tP_0}\|_{1\to 1} \leq 1$. Since the same is true in $L^2 \to L^2$, (15) follows by duality and interpolation. (16) follows from Proposition 1 and (15). □

The free KFP operator

To study the full KFP operator P, we want to treat the $W = -\nabla_x V(x) \cdot \nabla_v$ as perturbation and need some more estimates for e^{-tP_0}.

Proposition 3

Let $k \in \mathbb{N}$. The following estimates are true for the free KFP equation:

$$\|\langle v\rangle^k e^{-tP_0}\|_{1\to\infty} + \|\langle D_v\rangle^k e^{-tP_0}\|_{1\to\infty} \leq \frac{C}{(\gamma(t))^{\frac{n}{2}}}\left(1 + t^{-\frac{k}{2}}\right) \tag{17}$$

and for any $p \in [1, \infty]$,

$$\|\langle v\rangle^k e^{-tP_0}\|_{p\to p} + \|\langle D_v\rangle^k e^{-tP_0}\|_{p\to p} \leq C\left(1 + t^{-\frac{k}{2}}\right) \tag{18}$$

for $t > 0$.

The free KFP operator

Proof. $(L^1 - L^\infty)$. By the upper bound

$$0 \leq K(v, v', t) \leq \frac{1}{(4\pi\theta(t))^{\frac{n}{2}}} e^{-\frac{\cosh^2(t)-1}{2\sinh(2t)} v^2},$$

one obtains

$$\begin{aligned} & \|\langle v\rangle^k e^{-tP_0} f(\cdot, v)\|_{L^\infty_x} \\ \leq \quad & \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \sup_{v,v'} \langle v\rangle^k K(v, v', t) \|f\|_1 \\ \leq \quad & \frac{C}{(\gamma(t))^{\frac{n}{2}}} (1 + t^{-\frac{k}{2}}) \|f\|_1, \quad t > 0. \end{aligned}$$

□

The free KFP operator

By duality and interpolation, one obtains

Corollary 4

For $1 \leq p \leq q \leq \infty$ *and for any* $k \in \mathbb{N}$*, one has*

$$\|\langle v\rangle^k e^{-tP_0}\|_{p\to q} + \|\langle D_v\rangle^k e^{-tP_0}\|_{p\to q} \leq \frac{C}{(\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}}\left(1+t^{-\frac{k}{2}}\right), \quad (19)$$

and

$$\|e^{-tP_0}\langle v\rangle^k\|_{p\to q} + \|e^{-tP_0}\langle D_v\rangle^k\|_{p\to q} \leq \frac{C}{(\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}}\left(1+t^{-\frac{k}{2}}\right), \quad (20)$$

for $t > 0$.

Global-in-time estimates for e^{-tP}

Set $P = P_0 + W$ with $W = -\nabla_x V(x) \cdot \nabla_v$. Under the condition $\rho \geq -1$, W is relatively bounded perturbation of P_0 with relative bound 0 and P is closed with $D(P) = D(P_0)$. Since

$$e^{-tW} f(x, v) = f(x, v + t\nabla_x V(x)),$$

e^{-tP_0} and e^{-tW} are strongly continuous semigroups of contractions in L^p, $1 \leq p < \infty$. By theorem on perturbation of semigroup of contractions, e^{-tP} is a strongly continuous semigroup of contractions in L^p, $p \in [1, \infty[$.

We are interested in e^{-tP} when it is regarded as map from L^p to L^q, $q > p$.

Short-time estimates for e^{-tP}

Theorem 3

Let $n \geq 1$ and (3) be satisfied with $\rho \geq -1$. Then one has for $1 \leq p < q \leq \infty$

$$\|e^{-tP}\|_{p\to q} \leq \frac{C}{\gamma(t)^{\frac{n}{2p}(1-\frac{p}{q})}}, \quad t \in]0,1]. \tag{21}$$

Short-time estimates for e^{-tP}

Proof. One uses Duhamel's formula

$$e^{-tP} = e^{-tP_0} + \int_0^t e^{-(t-s)P_0} W e^{-sP} ds. \tag{22}$$

We apply (22) successively for $q > p$ and q near p such that

$$\int_0^t \|e^{-(t-s)P_0} W\|_{p \to q} ds < \infty$$

Let $\alpha(p, q) = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.

Short-time estimates for e^{-tP}

Take

$$1 = p_1 < p_2 < \cdots < p_{k-1} < p_k = 2 \quad \text{and} \quad \frac{1}{p_j} - \frac{1}{p_{j+1}} < \frac{1}{4n}.$$

Writing e^{-tP} as $(e^{-\frac{t}{k}P})^k$, one obtains

$$\begin{aligned} \|e^{-tP}\|_{1\to 2} &\leq \|e^{-\frac{t}{k}P}\|_{p_1\to p_2} \cdots \|e^{-\frac{t}{k}P}\|_{p_{k-1}\to 2} \\ &\leq C\gamma(t)^{-\alpha(1,p_2)-\cdots-\alpha(p_{k-1},2)} \\ &= C\gamma(t)^{-\alpha(1,2)} \end{aligned}$$

for $t \in]0,1]$. This proves (21) for $p = 1$ and $q = 2$. The general case follows by duality and interpolation. $\square$

Large-time estimate for e^{-tP}

For $t \geq 1$ large, we prove the following

Theorem 4

Assume $n = 3$ and that (3) is satisfied with $\rho > 1$. One has for $1 \leq p < q \leq \infty$

$$\|e^{-tP}\|_{p \to q} \leq Ct^{-\frac{3}{2p}(1-\frac{p}{q})} \tag{23}$$

for $t \in [1, \infty[$.

To prove Theorem 4, we use an earlier result proven by stationary method. The condition $n = 3$ is needed in low-energy spectral analysis of P.

Large-time estimate for e^{-tP}

Theorem 5 (W., CMP (2015))

Let $n = 3$ *and* $\rho > 1$. *For* $0 < r < s$ *and* $r \leq \frac{3}{2}$, *one has*

$$\|e^{-tP}\|_{\mathcal{L}^{2,s} \to \mathcal{L}^{2,-s}} \leq C \langle t \rangle^{-r}, t \geq 0. \tag{24}$$

Here $\mathcal{L}^{2,s} = L^2(\mathbb{R}^6_{x,v}, \langle x \rangle^{2s} dxdv)$.

The main task to prove (24) is to show that the resolvent of KFP $R(z) = (P - z)^{-1}$ admits an asymptotic expansion in appropriately weighted spaces :

$$R(z) = A_0 + \frac{iz^{\frac{1}{2}}}{4\pi} \langle \mathfrak{m}, \cdot \rangle \mathfrak{m} + O(|z|^{\frac{1}{2}+\epsilon})$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$ with $|z|$ small.

Large-time estimate for e^{-tP}

To prove Theorem 4, write

$$e^{-tP} = e^{-tP_0} + I(t) + J(t) \tag{25}$$

where

$$I(t) = \int_0^t e^{-(t-s)P_0} W e^{-sP_0}\, ds, \tag{26}$$

$$J(t) = \int_0^t \int_0^s e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0}\, d\tau ds. \tag{27}$$

Large-time estimate for e^{-tP}

Decompose $I(t) = I_1(t) + I_2(t)$ and $J(t) = J_1(t) + J_2(t)$ where

$$I_1(t) = \int_0^{\frac{t}{2}} e^{-(t-s)P_0} W e^{-sP_0} \, ds,$$

$$I_2(t) = \int_{\frac{t}{2}}^{t} e^{-(t-s)P_0} W e^{-sP_0} \, ds,$$

$$J_1(t) = \int_0^{\frac{t}{2}} \int_0^{s} e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} \, d\tau ds,$$

$$J_2(t) = \int_{\frac{t}{2}}^{t} \int_0^{s} e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} \, d\tau ds$$

Large-time estimate for e^{-tP}

$I_1(t)$ and $I_2(t)$ can be treated by the results for e^{-tP_0}. One has

$$\|I(t)\|_{1\to\infty} \leq Ct^{-\frac{3}{2}}, \quad \text{for } t \geq 1. \tag{28}$$

To treat $J(t)$, we use Theorem 5 on large-time estimate of e^{-tP} in weighted spaces.

Large-time estimate for e^{-tP}

Consider

$$J_1(t) = \int_0^{\frac{t}{2}} \int_0^{s} e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} \, d\tau ds,$$

where $W = -\nabla_x V(x) \cdot \nabla_v$. Then

- The cases $\tau \sim 0$ and $s - \tau \sim 0$ can be treated separately.

Large-time estimate for e^{-tP}

Consider

$$J_1(t) = \int_0^{\frac{t}{2}} \int_0^s e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0}\, d\tau ds,$$

where $W = -\nabla_x V(x) \cdot \nabla_v$. Then

- The cases $\tau \sim 0$ and $s - \tau \sim 0$ can be treated separately.
- $\|e^{-(t-s)P_0} \nabla_v\|_{1\to\infty} = O(t^{-\frac{3}{2}})$ for $s \le \frac{t}{2}$.

Large-time estimate for e^{-tP}

Consider

$$J_1(t) = \int_0^{\frac{t}{2}} \int_0^{s} e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0}\, d\tau ds,$$

where $W = -\nabla_x V(x) \cdot \nabla_v$. Then

- The cases $\tau \sim 0$ and $s - \tau \sim 0$ can be treated separately.
- $\|e^{-(t-s)P_0} \nabla_v\|_{1\to\infty} = O(t^{-\frac{3}{2}})$ for $s \leq \frac{t}{2}$.
- For $p > 3$ and close to 3, $s \to \|\nabla_v e^{-sP_0}\|_{1\to p}$ is integrable in $s \in [1, \infty[$.

Large-time estimate for e^{-tP}

We are left with the term $\nabla_x V e^{-\tau P} \nabla_x V : L^p \to L^1$ $p > 3$ close to 3 and $\tau > 1$. One has

- $\nabla_x V \in L^r(\mathbb{R}^3$ for $r > \frac{3}{1+\rho}$.

Large-time estimate for e^{-tP}

We are left with the term $\nabla_x V e^{-\tau P} \nabla_x V : L^p \to L^1$ $p > 3$ close to 3 and $\tau > 1$. One has

- $\nabla_x V \in L^r(\mathbb{R}^3$ for $r > \frac{3}{1+\rho}$.
-
$$\nabla_x V e^{-\delta P_0} : L^p \to \mathcal{L}^{2,\rho+\frac{1}{2}-\epsilon} \quad \text{and} \quad e^{-\delta P_0} \nabla_x V : \mathcal{L}^{2,-(\rho-\frac{1}{2}-\epsilon)} \to L^1$$

  are bounded.

Large-time estimate for e^{-tP}

Recall that

$$\|e^{-\tau P}\|_{\mathcal{L}^{2,\rho+\frac{1}{2}-\epsilon}\to\mathcal{L}^{2,-(\rho+\frac{1}{2}-\epsilon)}} = O(\langle\tau\rangle^{-(\rho-\frac{1}{2})+2\epsilon}).$$

One has

$$\|J_1(t)\|_{1\to\infty} \leq Ct^{-\frac{3}{2}} \int_0^{\frac{t}{2}} \int_1^s \langle\tau\rangle^{-(\rho-\frac{1}{2})+2\epsilon} \langle s-\tau\rangle^{-\frac{3(p-1)}{2p}} d\tau ds$$

for $t \geq 1$.

The same is also true for $J_2(t)$.

Large-time estimate for e^{-tP}

This proves

$$\|J(t)\|_{L^1\to L^\infty} \le \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } \rho > \frac{3}{2} \\ C_\epsilon t^{-\rho+\epsilon}, & \text{if } 1 < \rho \le \frac{3}{2}. \end{cases} \tag{29}$$

Consequently, Theorem 4 is proved if $\rho > \frac{3}{2}$. For $1 < \rho \le \frac{3}{2}$, one obtains

$$\|e^{-tP}\|_{L^p\to L^q} \le C_\epsilon t^{-\rho(\frac{1}{p}-\frac{1}{q})+\epsilon}, \tag{30}$$

for $t > 1$ and $1 \le p < q \le \infty$.

The case $1 < \rho \le \frac{3}{2}$ can be proved by a boost-up argument, making use of (30) instead of $\mathcal{L}^{2,s} - \mathcal{L}^{2,-s}$ estimate of e^{-tP}.

A remark on one dimensional case

Remark

Let $n = 1$ and $\rho > 4$. One has, $s > \frac{5}{2}$,

$$\|e^{-tP}\|_{\mathcal{L}^s \to \mathcal{L}^{-s}} \leq C\langle t\rangle^{-\frac{1}{2}}, t \geq 1. \tag{31}$$

The method used in the proof of Theorem 4 does not give any decay of e^{-tP} in $L^1 - L^\infty$ for t large. For example, for the term $I_1(t)$, our method gives

$$\|I_1(t)\|_{1\to\infty} \leq C\left(t^{-\frac{1}{2}} + \int_0^{\frac{t}{2}} \langle t-s\rangle^{-\frac{1}{2}}\langle s\rangle^{-\frac{1}{2}} ds\right), \quad t \geq 1.$$

The last integral does not decay as $t \to \infty$.